Peaks, Descents, and Pattern Avoidance

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Outline

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Permutations, Descents, and Peaks

- A permutation of length n (or, n-permutation) $\pi = \pi_1\pi_2\cdots\pi_n$ is a linear ordering of $[n] = \{1, 2, \ldots, n\}$.
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Example

Let $\pi = 6317254$. The descents of π are 6, 3, 7, and 5, and the peaks of π are 7 and 5, so $\mathrm{des}(\pi) = 4$ and $\mathrm{pk}(\pi) = 2$.

Eulerian Polynomials and Peak Polynomials

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Example

Permutations of length 3:

123, 132, 213, 231, 312, 321

Thus, $A_3(t) = 1 + 4t + t^2$ and $P_3(t) = 4 + 2t$.

γ-Positivity

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Theorem (Foata and Schützenberger 1970)

The Eulerian polynomials $A_n(t)$ are γ-positive.

Stembridge's Magical Formula

Theorem (Stembridge 1997)

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n-1} P_n\left(\frac{4t}{(1+t)^2}\right)$$

- Equivalently,

$$P_n(t) = \left(\frac{2}{1+v}\right)^{n-1} A_n(v)$$

where $v = \frac{2}{t}(1-\sqrt{1-t}) - 1$.

Brändén's Generalization

- For $\Pi \subseteq \mathfrak{S}_n$, let

$$A(\Pi; t) := \sum_{\pi \in \Pi} t^{\mathrm{des}(\pi)} \qquad \text{and} \qquad P(\Pi; t) := \sum_{\pi \in \Pi} t^{\mathrm{pk}(\pi)}.$$

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- We define the modified Foata–Strehl action, state Brändén's results, and generalize them further.

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Peaks, Valleys, Double Ascents, and Double Descents

- Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation.
- We say that π_i is a peak of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$.
- We say that π_i is a valley of π if $\pi_{i-1} > \pi_i < \pi_{i+1}$.
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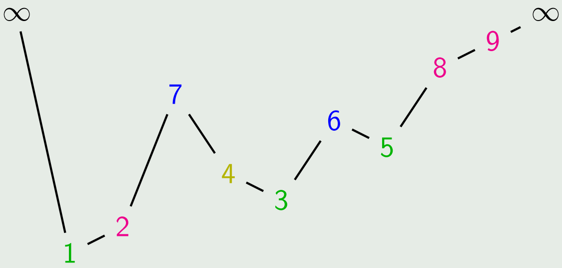
- We will work with peaks, valleys, double ascents, and double descents of the word $\check{\pi} := \infty\pi\infty$.

Setting Up The Modified Foata–Strehl Action

Example

Take $\pi = 127436589$.



(Color coding: peaks, valleys, double ascents, double descents.)

Setting Up The Modified Foata–Strehl Action (cont.)

- Let $x \in [n]$ be a double ascent or double descent of $\check{\pi}$. Then let $\mathtt{hop}_x(\pi)$ be the permutation obtained from π by letting x "valley-hop".

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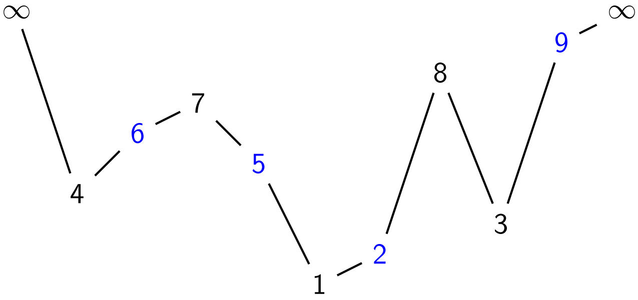

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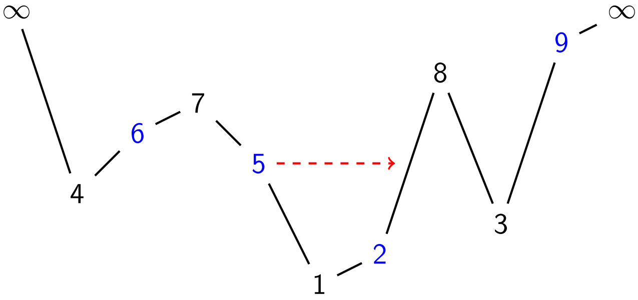

- $\mathrm{hop}_5(467{\color{magenta}5}12839) = 46712{\color{magenta}5}839$

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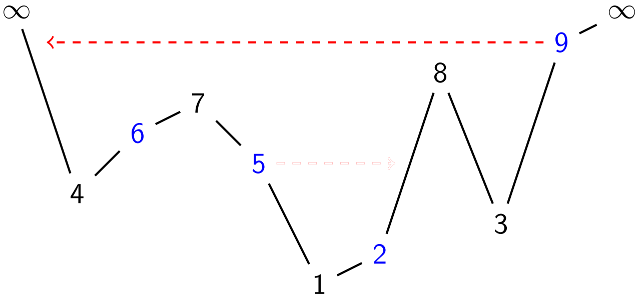

- $\mathtt{hop}_5(467512839) = 467125839$
- $\mathtt{hop}_9(467512839) = 946751283$

Setting Up The Modified Foata–Strehl Action (cont.)

- Let $\pi \in \mathfrak{S}_n$ and let $x \in [n]$. Define φ_x by

$$\varphi_x(\pi) = \begin{cases} \mathtt{hop}_x(\pi), & \text{if } x \text{ is a double ascent or double descent of } \check{\pi}, \\ \pi, & \text{if } x \text{ is a peak or valley of } \check{\pi}; \end{cases}$$

  these are involutions that commute with each other.

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- Also, for $S \subseteq [n]$, define φ_S by

$$\varphi_S(\pi) = \Big(\prod_{x \in S} \varphi_x\Big)(\pi);$$

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- The group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the involutions φ_S; this action is called the modified Foata–Strehl action (or valley-hopping), abbreviated MFS-action.

Brändén's Theorem

Theorem (Brändén 2008)

Suppose that $\Pi \subseteq \mathfrak{S}_n$ is invariant under the MFS-action. Then the polynomial $A(\Pi; t)$ is γ-positive, and

$$A(\Pi; t) = \left(\frac{1+t}{2}\right)^{n-1} P\left(\Pi; \frac{4t}{(1+t)^2}\right).$$

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Theorem (Brändén 2008)

The set of k-stack-sortable n-permutations is invariant under the MFS-action for any k.

- The set of 1-stack-sortable n-permutations is equal to $\mathrm{Av}_n(231)$, whose descent polynomial is the nth Narayana polynomial.

A Refinement of Brändén's Theorem

- Given a set Π of permutations, let

$$P(\Pi; y, t) := \sum_{\pi \in \Pi} y^{\mathrm{pk}(\pi)} t^{\mathrm{des}(\pi)}.$$

Theorem (Z. 2017+)

Suppose that $\Pi \subseteq \mathfrak{S}_n$ for $n \geq 1$ is invariant under the MFS-action. Then

$$A(\Pi; t) = \left(\frac{1+yt}{1+y}\right)^{n-1} P\left(\Pi; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right).$$

Equivalently,

$$P(\Pi; y, t) = \left(\frac{1+u}{1+uv}\right)^{n-1} A(\Pi; v)$$

where $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}$ *and* $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$.

Idea of Proof

- Fix $\sigma \in \Pi$. Then

$$\Big(\sum_{\pi \in \mathrm{Orb}(\sigma)} t^{\mathrm{des}(\pi)}\Big)(1+y)^{\mathrm{dasc}(\check{\sigma})+\mathrm{ddes}(\check{\sigma})} = \sum_{\pi \in \mathrm{Orb}(\sigma)} (1+yt)^{\mathrm{dasc}(\check{\pi})}(y+t)^{\mathrm{ddes}(\check{\pi})}t^{\mathrm{pk}(\check{\pi})};$$

both sides count permutations in $\mathrm{Orb}(\sigma)$ with a marked subset of double ascents and double descents, where y is weighting the number of marked letters and t is weighting the descent number.

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- We can express $\mathrm{pk}(\check{\pi})$, $\mathrm{dasc}(\check{\pi})$, and $\mathrm{ddes}(\check{\pi})$ in terms of $\mathrm{pk}(\pi)$ and $\mathrm{des}(\pi)$, which gives us

$$\sum_{\pi\in\mathrm{Orb}(\sigma)} t^{\mathrm{des}(\pi)} = \sum_{\pi\in\mathrm{Orb}(\sigma)} \frac{(1+yt)^{n-\mathrm{pk}(\pi)-\mathrm{des}(\pi)-1}(y+t)^{\mathrm{des}(\pi)-\mathrm{pk}(\pi)}t^{\mathrm{pk}(\pi)}}{(1+y)^{n-2\,\mathrm{pk}(\pi)-1}}.$$

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- Sum over all orbits contained in Π.

Similar Results

- By taking $\Pi = \mathfrak{S}_n$, we get an analogous expression for the $(\mathrm{pk}, \mathrm{des})$ polynomials

$$P_n(y,t) := \sum_{\pi \in \mathfrak{S}_n} y^{\mathrm{pk}(\pi)} t^{\mathrm{des}(\pi)}$$

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- Similar results for other permutation statistics—as well as q-analogues and type B analogues—can be found in the following paper:

  Yan Zhuang. Eulerian polynomials and descent statistics. *Adv. in Appl. Math.* 90: 86–144, 2017.

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- For simplicity, we are only considering pattern sets Σ with patterns that are all of the same length.

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Theorem (Zhou and Z. 2017+)

Let Σ be an orbit of the MFS-action. Then $\mathrm{Av}_n(\Sigma)$ is MFS-invariant for every $n \geq 1$ if and only if Σ is one of the following:

$\{132\}$	$\{3412, 3421\}$
$\{231\}$	$\{12534, 12543, 21534, 21543\}$
$\{1243, 2143\}$	$\{13524, 13542, 31524, 31542\}$
$\{1342, 3142\}$	$\{14523, 14532, 41523, 41532\}$
$\{2341, 3241\}$	$\{23514, 23541, 32514, 32541\}$
$\{1423, 1432\}$	$\{24513, 24531, 42513, 42531\}$
$\{2413, 2431\}$	$\{34512, 34521, 43512, 43521\}$

Pattern Sets Containing a Prescribed Pattern

Theorem (Zhou and Z. 2017+)

If $\sigma \in \mathfrak{S}_k$ has at least one peak, then there exists $\Sigma \subsetneq \mathfrak{S}_k$ containing σ such that $\mathrm{Av}_n(\Sigma)$ is MFS-invariant for every $n \geq 1$.

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Example

Let $\sigma = 13254$. Then the first construction yields

$$\Sigma = \{13254, 14253, 15243\},$$

whereas the second construction yields

$$\Sigma = \{13254, 12354, 21354, 23154, 31254, 32154\}.$$

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THANK YOU!