

# Packing Densities: An Update

(joint work with Jakub Sliáčan)

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Permutation Patterns 2017

June 27, 2017

# Packing Densities

Let  $\sigma$  be a permutation of size  $k$  (the "pattern"),  
and let  $\pi$  be any permutation.

The packing density of  $\sigma$  in  $\pi$  is

$\delta(\sigma, \pi)$  = Probability that a random  $k$ -subsequence  
of  $\pi$  is order-isomorphic to  $\sigma$ .

Then:

$$\delta(\sigma, n) = \max_{\pi \in S_n} \delta(\sigma, \pi)$$

and

$$\delta(\sigma) = \lim_{n \rightarrow \infty} \delta(\sigma, n).$$

We want to know  $\delta(\sigma)$  for various patterns  $\sigma$ .

# Packing Densities with Permutons

A *permuton*  $\mu$  is a probability measure on  $[0,1]^2$  with uniform marginals.

The *packing density* of  $\sigma$  in  $\mu$  is

$\delta(\sigma, \mu) =$  Probability that  $k$  points drawn independently from  $\mu$  are order-isomorphic to the graph of  $\sigma$ .

Then

$$\delta(\sigma) = \max_{\mu} \delta(\sigma, \mu)$$

and a permuton achieving this maximum is an *optimizer* for  $\sigma$ . This definition of  $\delta(\sigma)$  agrees with the previous definition.

(Alternatively: If you like,  $\mu$  is just a permutation of size  $n$ , and we only care about the case of  $n \rightarrow \infty$ .)

## Packing Densities for Patterns of Size 3, 4

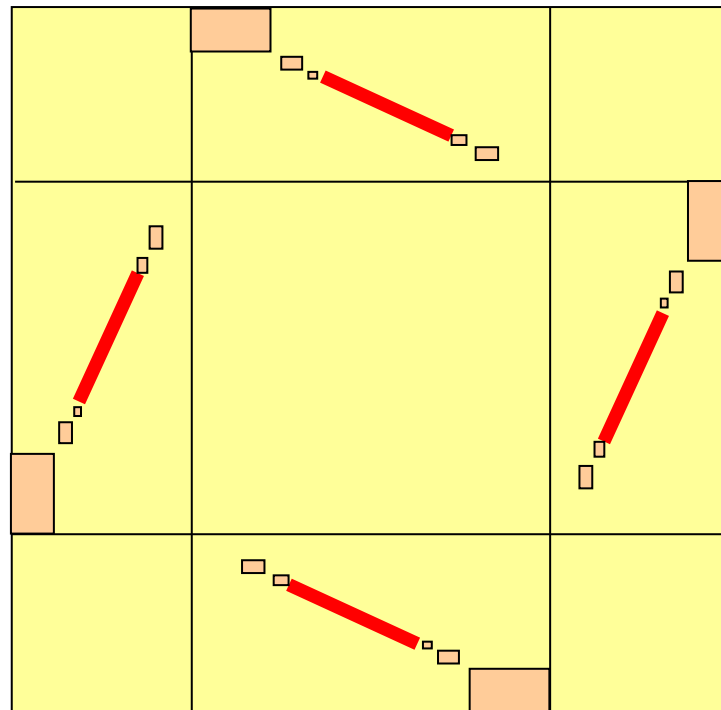
$\sigma$	Upper Bound	Lower Bound
123	1	
132	$2\sqrt{3} - 3 \approx 0.464$	
1234	1	
1243	3/8	
2143	3/8	
1432	algebraic, $\approx 0.4236$	
1324	0.2440545239294233295868733...	
2413	$\approx 0.104724$	$\approx 0.104780$
1342	$\approx 0.198836597$	$\approx 0.198837287$

For 1324, the only update is that we can now compute a hundred digits or more. We still don't know whether the packing density is algebraic.

# Packing 2413

Upper Bound  $\approx 0.104780$  is from Flag Algebra, JS PP 2016.

Lower Bound  $\approx 0.104724$  is from Presutti and WRS, PP 2007:

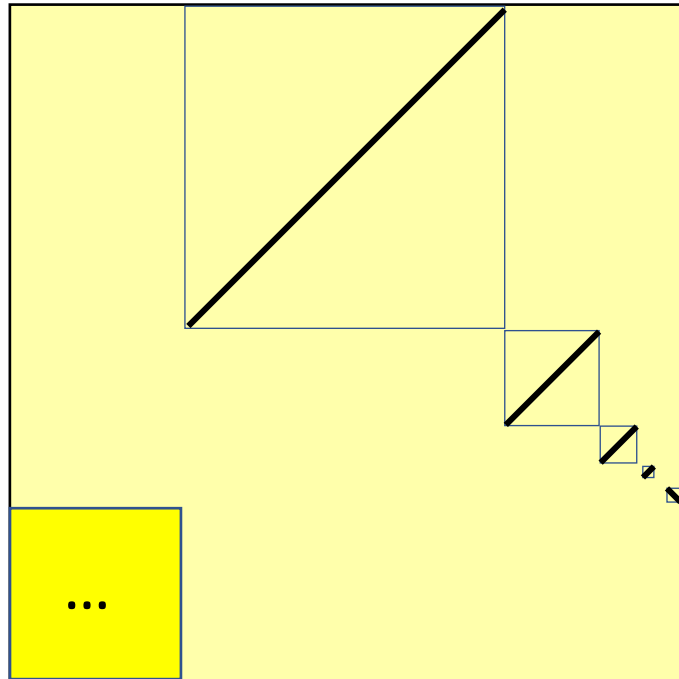


Neither the upper bound nor the lower bound pretends to be exact.

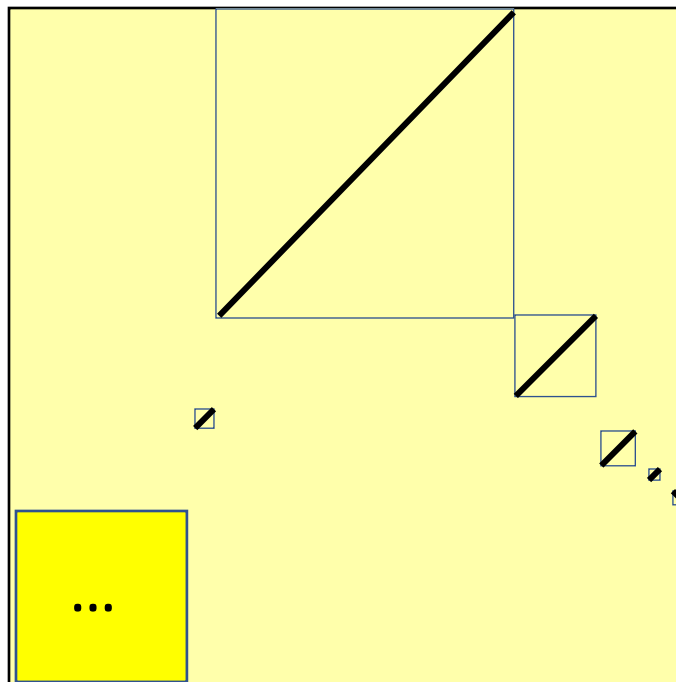
# Packing 1342

Our main update this year is for  $\delta(1342)$ .

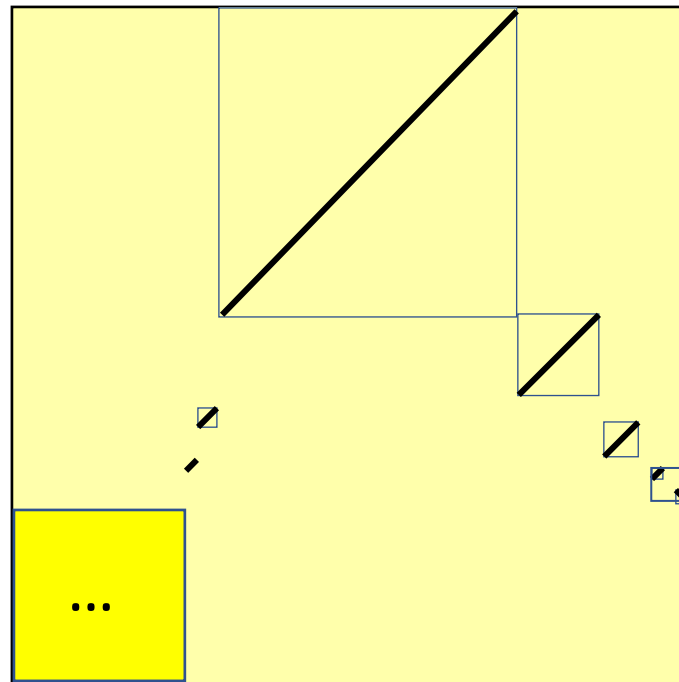
The previous lower bound was  $\approx 0.1965796$ , based on this construction by Batkeyev.



But we can do better by adding a tiny box:



And then, why not another one?



But that's the end of it. Adding more boxes in the middle doesn't help; this is the best we can do with this theme. And since there are only seven boxes, the optimum box sizes (and the packing density) must be algebraic.



Box sizes (left to right)

$a_1 = 0.2174127723536347308692444843$

$a_2 = 0.0170598057899242722740620549$

$a_3 = 0.0516101402487892270230230972$

$a_4 = 0.4340722809873864994312953007$

$a_5 = 0.1479895625950390496250611829$

$a_6 = 0.0764457255805656971383351365$

$a_7 = 0.0554097124446605236389787433$

This construction gives the lower bound  $\approx 0.198836597$  for  $\delta(1342)$ .

The current upper bound (Flag Algebra, JS PP2016) is  $\approx 0.198837287$ .

# The Density Vector

For a permutation  $\mu$ , consider its "density vector" with all of its size-3 pattern densities:

$$X(\mu) = ( \delta(123, \mu), \delta(132, \mu), \delta(213, \mu), \delta(231, \mu), \delta(312, \mu), \delta(321, \mu) ).$$

I'm tired of typing the  $\mu$ , so let  $\mu$  be understood and use this notation:

$$X(\mu) = ( \delta_{123}, \delta_{132}, \delta_{213}, \delta_{231}, \delta_{312}, \delta_{321} ) \in \mathbb{R}^6.$$

What can this vector be?

This might be a hard problem. The set of possible vectors is not convex. It has cross sections that aren't even simply connected.

But we do know some things. The six entries add to 1. And...

# Linear Constraints on the Density Vector

...the entries are subject to some linear constraints.

$$\delta_{132} \leq 0.464\dots \quad (\text{packing density of } 132)$$

$$\delta_{123} + \delta_{321} \geq 1/4 \quad (\text{monotone triples})$$

$$\delta_{213} + \delta_{312} \leq 1/2 \quad (\text{"V-shaped things"})$$

How can we find more of these ?

Specifically, we want formulas of the form

$$\alpha_{123} \delta_{123} + \alpha_{132} \delta_{132} + \alpha_{213} \delta_{213} + \alpha_{231} \delta_{231} + \alpha_{312} \delta_{312} + \alpha_{321} \delta_{321} \geq 0.$$

For example, suppose we knew (from some magical machine) that

$$3 \delta_{123} + (-2\sqrt{3}) \delta_{132} + 3 \delta_{213} + (1 - \sqrt{3}) \delta_{231} + (1 - \sqrt{3}) \delta_{312} + 3 \delta_{321} \geq 0.$$

Then with about two lines of algebra, we can conclude that

$$\delta_{132} \leq (2\sqrt{3}-3) - (6-3\sqrt{3}) (\delta_{231} + \delta_{312}).$$

That's an upper bound for the packing density! And it's better (because of the extra terms) than the standard version. And---thanks to that magical machine---it is almost free.

# Flag Algebra

A flag  $F$  of size  $(k, t)$  is a permutation of size  $k$ , together with a designated subsequence of size  $t$ .

For example:

5 (2) 3 1 (4)

The "outer permutation" is 5 2 3 1 4.

The designated subsequence is 2 4.

That defines the "inner permutation" 1 2. If the inner permutation is  $\tau$ , then  $F$  is called a  $\tau$ -flag. So this is a 12-flag.

There are exactly four flags of size  $(2,1)$ . They are all "1-flags."

$F_1$	$F_2$	$F_3$	$F_4$
(1) 2	1 (2)	(2) 1	2 (1)

The density of a flag  $F$  in a permutation  $\mu$  is

$\delta(F, \mu) =$  Probability that  $k$  independent draws from  $\mu$ ,  
with  $t$  of them randomly designated as a subsequence,  
are order-isomorphic to the graph of  $F$ .

We also need to know about *joint appearances* of two flags  $F_1$  and  $F_2$ , both of which have size  $(k, t)$  and the same inner permutation  $\tau$ .

Suppose a permutation  $\pi$  of size  $(2k - t)$  can be divided into two subsequences of length  $k$ , which are isomorphic to  $F_1$  and  $F_2$  respectively, and which overlap only on their images of  $\tau$ . That's a joint appearance.

For example,  $2\ 1\ 3$  is a joint appearance of  $F_4 = 2\ 1$  and  $F_1 = 1\ 2$ .

The same pattern  $213$  can be interpreted as a joint appearance of  $F_1$  and  $F_3$ , or of  $F_2$  and  $F_2$ .

Each of the other size-3 patterns can be interpreted as a joint appearance in three ways, also.

Fact:

$$\delta(F_1, \mu) \delta(F_2, \mu) = \sum_{\pi} (\delta(\pi, \mu)) \text{ (Number of ways to represent } \pi \text{ as a joint appearance of } F_1 \text{ and } F_2 \text{ )}.$$

P.S. This is mostly true but oversimplified. Prove it for a particular occurrence of  $\tau$  in  $\mu$ , then aggregate; discard factors that don't depend on  $\pi$  as they won't matter in the end.

# The Flag Algebra Machine

Now let's select a collection of flags with the same size and same  $\tau$ .  
Today, we will use the four flags of size (2,1).

And, let's pick ANY four real numbers  $q_1, q_2, q_3, q_4$ .

Then:

$$\begin{aligned} 0 &\leq (q_1 \delta(F_1, \mu) + q_2 \delta(F_2, \mu) + q_3 \delta(F_3, \mu) + q_4 \delta(F_4, \mu))^2 \\ &= \sum_{i,j} q_i q_j \delta(F_i, \mu) \delta(F_j, \mu) \\ &= \sum_{i,j} q_i q_j \sum_{\pi} \delta(\pi, \mu) (\text{Number of ways to represent } \pi \text{ as} \\ &\quad \text{a joint appearance of } F_i \text{ and } F_j) \\ &= \sum_{\pi} (\sum_{i,j} q_i q_j (\text{Number of ways } \dots)) \delta(\pi, \mu) \\ &= \sum_{\pi} (\alpha_{\pi}) \delta(\pi, \mu). \quad \text{THAT'S WHAT WE WANTED.} \end{aligned}$$



# Very Concrete Results

For ANY choice of  $q_1, q_2, q_3, q_4$ , we have:

$$\begin{aligned} & (q_1^2 + q_1 q_2 + q_2^2) \delta(123, \mu) \\ + & (q_1^2 + q_2 q_3 + q_2 q_4) \delta(132, \mu) \\ + & (q_2^2 + q_1 q_3 + q_1 q_4) \delta(213, \mu) \\ + & (q_4^2 + q_2 q_3 + q_1 q_3) \delta(231, \mu) \\ + & (q_3^2 + q_1 q_4 + q_2 q_4) \delta(312, \mu) \\ + & (q_3^2 + q_3 q_4 + q_4^2) \delta(321, \mu) \geq 0. \end{aligned}$$

When  $q_1 = 1, q_2 = q_3 = 0, q_4 = -1$  we get:

$$\delta_{213} + \delta_{312} \leq 1/2 \quad (\text{"V-shaped things"})$$

When  $q_1 = 0, q_2 = \sqrt{3}, q_3 = q_4 = -1$  we get:

$$3 \delta_{123} + (-2\sqrt{3}) \delta_{132} + 3 \delta_{213} + (1 - \sqrt{3}) \delta_{231} + (1 - \sqrt{3}) \delta_{312} + 3 \delta_{321} \geq 0$$

(which gives us the packing density of 132).

Let  $\phi = \frac{1+\sqrt{5}}{2} = \text{"golden mean"} \approx 1.6218.$

When  $q_1 = -\phi, q_2 = \phi, q_3 = 1, q_4 = -1$  we get

$$(2\phi^2)(\delta_{123} + \delta_{321}) - (2\phi)(\delta_{213} + \delta_{312}) \geq 0$$

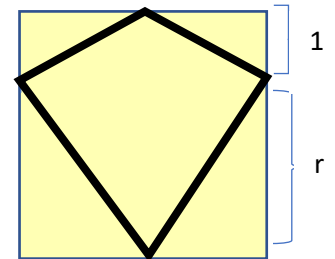
which simplifies to

$$(\delta_{213} + \delta_{312}) / (\delta_{123} + \delta_{321}) \leq \phi$$

or

$$\frac{\text{Number of V-shaped patterns}}{\text{Number of monotone patterns}} \leq \phi$$

P. S. 6/29/17: Andrew Elvey Price found this permuton.  
The V-shaped/Monotone ratio has a maximum value of  $\phi$ ,  
when the ratio  $r$  is  $\phi$ .



in any permuton or (in the limit) large permutation. Did we know that?