On Fishburn Numbers

Permutation Patterns, 29. 6. 2017

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The Plan

- Fishburn-enumerated objects
- Generating functions, asymptotics, congruences
- The Catalan connection

Part I

Fishburn-Enumerated Objects

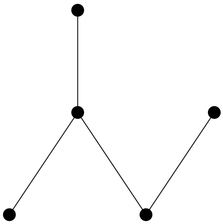
Interval Orders

Definition (Fishburn, 1970)

A partially ordered set $(\mathcal{P}, \prec)$ is an interval order if we can assign to every $x \in \mathcal{P}$ a (closed, bounded) interval I_x, such that the following holds:

$$x \prec y \iff \max(I_x) < \min(I_y).$$

The multiset $\{I_x;\ x \in \mathcal{P}\}$ is an interval representation of $\mathcal{P}$.



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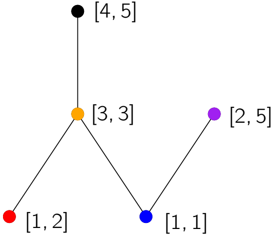

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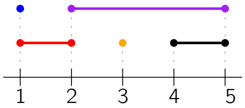

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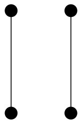

"**2+2**"

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Theorem (Fishburn)

A poset is an interval order iff it avoids $\mathbf{2}+\mathbf{2}$ *as induced subposet.*

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Fishburn Matrices

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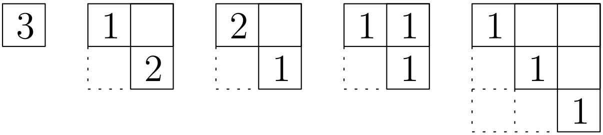
A Fishburn matrix is a square matrix $M = (m_{i,j})_{i,j=1}^{d}$ of nonnegative integers such that

- M is upper-triangular (i.e., $m_{i,j} = 0$ whenever $i > j$), and
- every row and every column of M has a nonzero entry.



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The weight of a matrix is the sum of its entries.

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad \begin{array}{cc} 1 & \\ & 2 \end{array} \qquad \begin{array}{cc} 2 & \\ & 1 \end{array} \qquad \begin{array}{cc} 1 & 1 \\ & 1 \end{array} \qquad \begin{array}{ccc} 1 & & \\ & 1 & \\ & & 1 \end{array}$$

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A matrix is primitive if all its entries are 0 or 1.

$$\begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

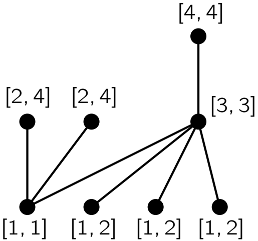
From Matrices to Interval Orders

Theorem (Fishburn, 1970's; Dukes–Parviainen, 2010)

There are f_n Fishburn matrices of weight n.

	1	2	3	4
1	1	3		
2				2
3			1	
4				1

$$\to \{[1,1],[1,2],[1,2],[1,2],[2,4],[2,4],[3,3],[4,4]\} \to$$



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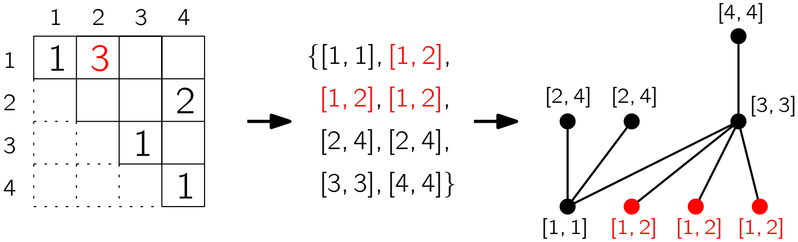

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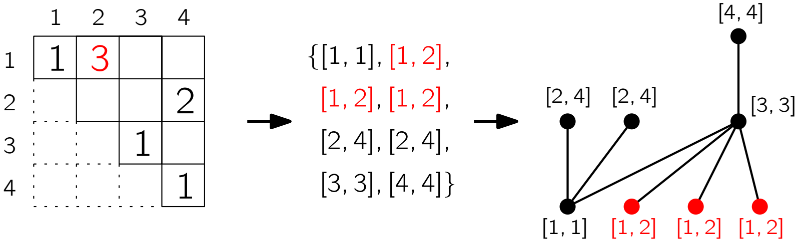
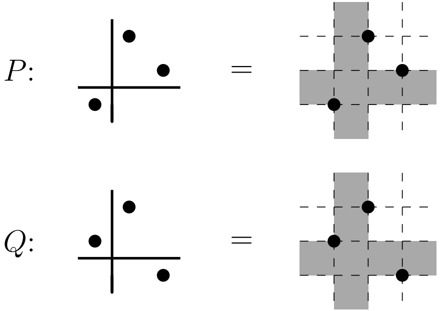


Theorem

Fishburn matrices of weight n, with first row of weight k and last column of weight ℓ correspond to interval orders of size n, with k minimal and ℓ maximal elements.

More Fishburn Objects: Avoiders of Vincular Patterns



Theorem (Bousquet-Mélou–Claesson–Dukes–Kitaev, 2009; Parviainen, 2009)

$|Av_n(P)| = |Av_n(Q)| = f_n$.

Part II

Generating Functions, Asymptotics, Congruences

The Generating Function

Notation

For $n \geq 0$, the notation $(a;q)_n$ denotes the product $(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$.

Theorem (Zagier, 2001; Bousquet-Mélou et al., 2009)

$$\sum_{n=1}^{\infty} f_n x^n = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \left(1-(1-x)^j\right) = \sum_{k=1}^{\infty} (1-x;1-x)_k \, .$$

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Deriving the Generating Function

- recall: f_n ... number of Fishburn matrices of weight n
- $f_{k,\ell}$ ... number of Fishburn matrices of weight $k+\ell$, with last column of weight ℓ ($k \geq 0$, $\ell \geq 1$)
- $F(x,y) = \sum_{k,\ell} f_{k,\ell} x^k y^\ell$ (hence $F(x,x) = \sum_{n \geq 1} f_n x^n$)
- $p_{k,\ell}$ ... number of primitive Fishburn matrices of weight $k+\ell$, with last column of weight ℓ
- $P(x,y) = \sum_{k,\ell} p_{k,\ell} x^k y^\ell$

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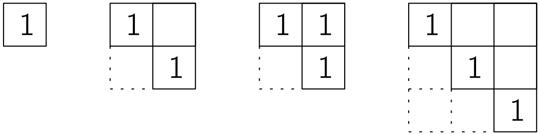

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$F(x,y) = P(\frac{x}{1-x}, \frac{y}{1-y})$.

- Goal: show that $P(x,y) = \sum_{n=1}^{\infty} \left(\frac{1}{1+y}; \frac{1}{1+x}\right)_n$ and therefore $F(x,y) = \sum_{n=1}^{\infty} (1-y; 1-x)_n$ and $F(x,x) = \sum_{n=1}^{\infty} (1-x; 1-x)_n$.

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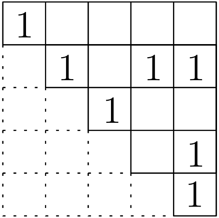
Idea: extend a primitive Fishburn matrix by adding a column.



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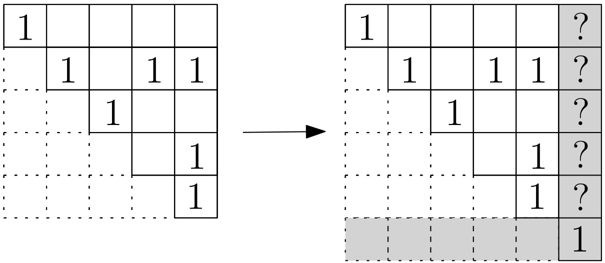

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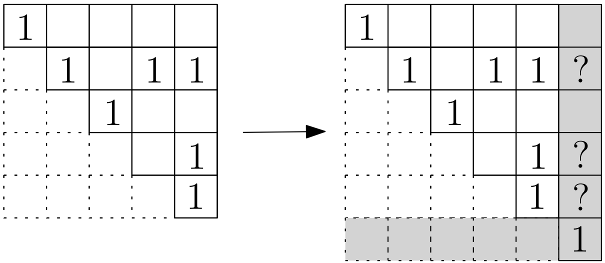

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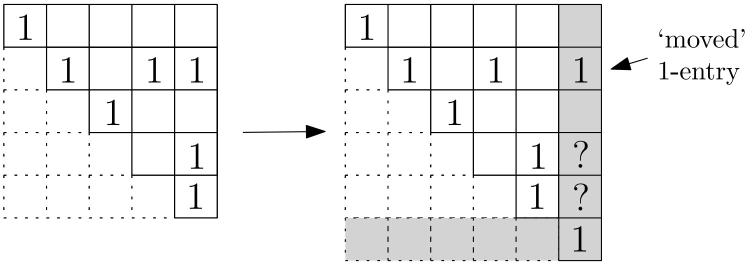

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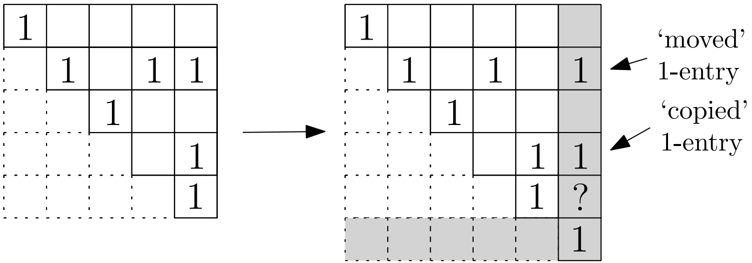

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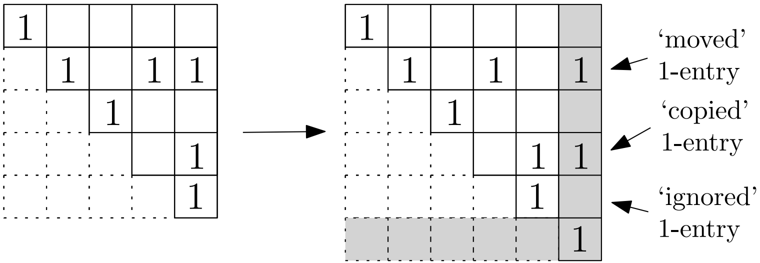

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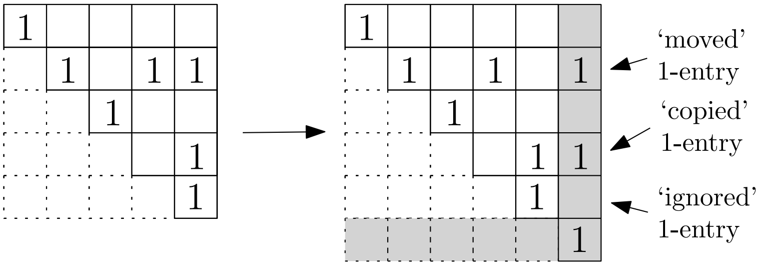

- $3^3 = 27$ possibilities, 1 of them is ‘bad’.
- In terms of gen. functions: $x^k y^\ell \to yx^k(x+y+xy)^\ell - yx^k y^\ell$.
- Hence: $P(x,y) = y + yP(x, x+y+xy) - yP(x,y)$,
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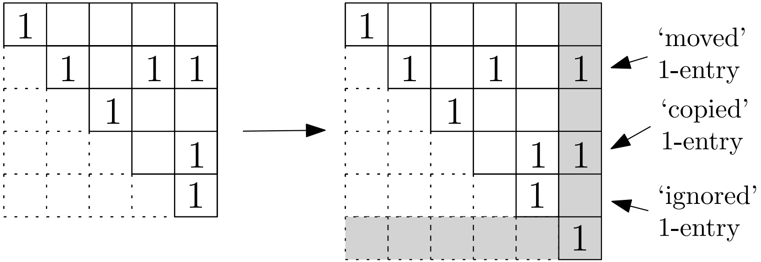

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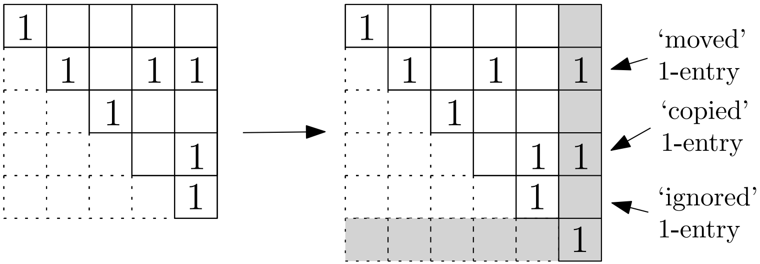

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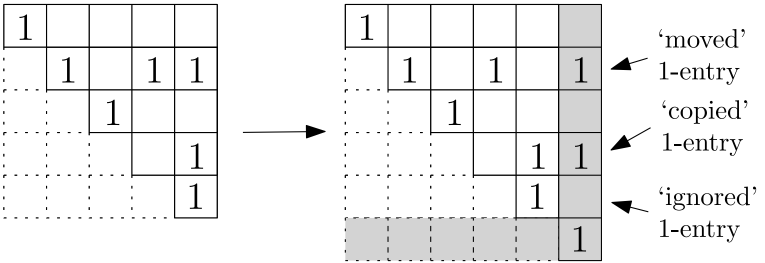

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Solving the Functional Equation

Recall: $P(x,y) = \left(1 - \frac{1}{1+y}\right) + \left(1 - \frac{1}{1+y}\right) P(x, x+y+xy)$

Note: the substitution $y \mapsto x + y + xy$ is equivalent to $(1+y) \mapsto (1+y)(1+x)$.

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&= \left(1 - \frac{1}{1+y}\right) + \left(1 - \frac{1}{1+y}\right)\left(1 - \frac{1}{(1+y)(1+x)}\right) + \\
&+ \left(1 - \frac{1}{1+y}\right)\left(1 - \frac{1}{(1+y)(1+x)}\right)\left(1 - \frac{1}{(1+y)(1+x)^2}\right) + \cdots
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Hence: $F(x,y) = \sum_{n\geq 1} (1-y; 1-x)_n$, and $F(x) = \sum_{n\geq 1} (1-x; 1-x)_n$.

Asymptotics

Theorem (Zagier, 2001)

$$f_n = n!\left(\frac{6}{\pi^2}\right)^n \sqrt{n}\left(\alpha + O\left(\frac{1}{n}\right)\right) \qquad \textit{with } \alpha = \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12}.$$

How to deduce the formula numerically:

- Compute f_n for $n \lesssim 1000$ (easy, since we know the GF)
- Observe $f_n \approx c^n n!$ for a constant c. Wanted: the value of c.
- Idea: define $r_n = \frac{f_{n+1}}{n f_n}$. Then $\lim_{n\to\infty} r_n = c$.
- Problem: $r_{1000} = 0.60823163\ldots$ is still far from $\frac{6}{\pi^2} = 0.60792710\ldots$.

Asymptotics

Theorem (Zagier, 2001)

$$f_n = n! \left(\frac{6}{\pi^2}\right)^n \sqrt{n} \left(\alpha + O\left(\frac{1}{n}\right)\right) \qquad \text{with } \alpha = \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12}.$$

How to deduce the formula numerically:

- Compute f_n for $n \lesssim 1000$ (easy, since we know the GF)
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- For a sequence $(a_n)_{n\in\mathbb{N}}$, let $\Delta(a_n)$ be the sequence $(a_{n+1} - a_n)_{n\in\mathbb{N}}$ ("the difference of (a_n)").
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- More generally, for fixed $k \in \mathbb{N}$, $\Delta^{(k)}(n^k r_n/k!) = c + O(\frac{1}{n^{k+1}})$.
- Set $k = 100$ and define $(t_n)_{n\in\mathbb{N}} = \Delta^{(k)}(n^k r_n/k!)$. Then $|t_{1000} - \frac{6}{\pi^2}| < 10^{-180}$, suggesting that $\frac{6}{\pi^2}$ is the limit of t_n, and therefore also of r_n.

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Congruences

- Sequence of $f_n \mod 5$, for $n = 0, \ldots, 99$:

$$1, 1, 2, 0, 0, 3, 2, 4, 0, 0, 3, 4, 3, 0, 0, 0, 2, 4, 0, 0, 3, 1, 2, 0, 0,$$
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- apparently, $f_{5k+3} \equiv f_{5k+4} \equiv 0$ and $f_{5k+2} \equiv 2f_{5k+1} \mod 5$.
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Theorem (Andrews–Sellers, 2014)

For any prime p that is a quadratic nonresidue mod 24 there is a $i \geq 1$ such that for every $k \geq 1$, we have

$$f_{pk-1} \equiv f_{pk-2} \equiv \cdots \equiv f_{pk-i} \equiv 0 \mod p.$$

- Generalized to congruences modulo prime powers [Straub, 2014].
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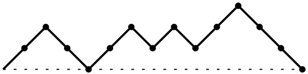
Open problem

Does any of the congruence properties of f_n have a combinatorial explanation?

The Catalan Connection

Dyck Paths and Their Statistics



- $\mathrm{ret}(D)$ … number of returns of D (i.e., down-steps reaching the x-axis)
- $\mathrm{tail}(D)$ … number of down steps following the last up-step.

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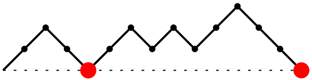

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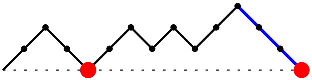

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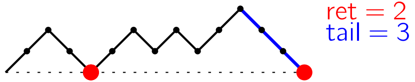

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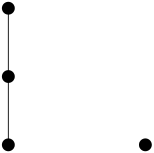
Theorem (Kreweras, 1970; Vaillé, 1997)

The statistics ret and tail have symmetric joint distribution on the set of Dyck paths of a given length.

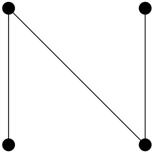
The Catalan–Fishburn Connection

Theorem (Kim–Roush, 1978; Disanto–Ferrari–Pinzani–Rinaldi, 2010)

The $(\mathbf{2}+\mathbf{2},\mathbf{N})$*-free posets of size* n*, as well as the* $(\mathbf{2}+\mathbf{2},\mathbf{3}+\mathbf{1})$*-free posets of size* n *are counted by Catalan numbers.*



$\mathbf{3}+\mathbf{1}$



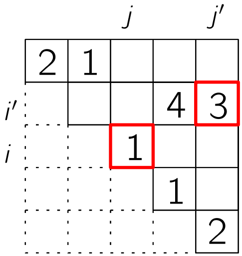
$\mathbf{N}$

Catalan Classes of Fishburn Matrices

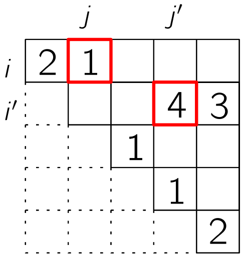
Definition

Let $M = (m_{ij})$ be a Fishburn matrix.

- NE-pair . . . a pair of nonzero cells m_{ij} and $m_{i'j'}$ such that $i > i'$ and $j < j'$.
- SE-pair . . . a pair of nonzero cells m_{ij} and $m_{i'j'}$ such that $i < i'$, $j < j'$ and $i' \leq j$.



NE-pair



SE-pair

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Definition

A Fishburn matrix is NE-free (or SE-free) if it has no NE-pair (or SE-pair, respectively).

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A Fishburn matrix is NE-free (or SE-free) if it has no NE-pair (or SE-pair, respectively).

Theorem (Dukes–J.–Kubitzke, 2011; J., 2015)

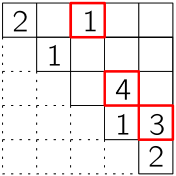
*(**2+2**,**N**)-free posets correspond to SE-free Fishburn matrices, (**2+2**,**3+1**)-free posets correspond to NE-free Fishburn matrices.*

Statistics of Fishburn Matrices

Definition

Let $M = (m_{ij})$ be a $k \times k$ Fishburn matrix.

- NE-extreme . . . a nonzero cell m_{ij} such that all the other cells in rows $1, \ldots, i$ and columns $j, \ldots, k$ are zeros.
- SE-extreme . . . a nonzero diagonal cell m_{ii} such that all the other cells in row i are zeros.
- $\mathrm{ne}(M), \mathrm{se}(M), \mathrm{lc}(M)$ . . . number of NE-extreme cells, number of SE-extreme cells, weight of the last column of M, respectively.



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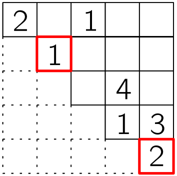

SE-extremes

Statistics of Fishburn Matrices

Definition

Let $M = (m_{ij})$ be a $k \times k$ Fishburn matrix.

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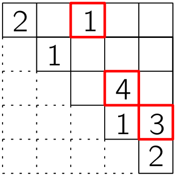

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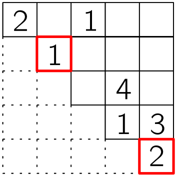

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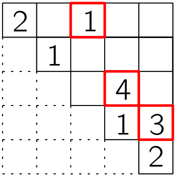

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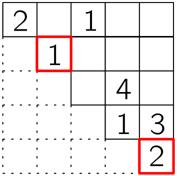

SE-extremes

Dyck Paths ∼ (**2**+**2**,**N**)-free Posets ∼ SE-free Matrices

Theorem

The above is a bijection from Dyck paths to SE-free Fishburn matrices. A path D of length $2n$ maps to a matrix M of weigth n with $\mathrm{ret}(D) = \mathrm{ne}(M)$ and $\mathrm{tail}(D) = \mathrm{lc}(M)$.

In particular, $\mathrm{ne}(\cdot)$ and $\mathrm{lc}(\cdot)$ have symmetric joint distribution on SE-free Fishburn matrices.

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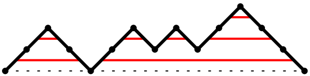

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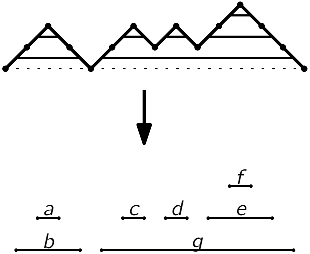

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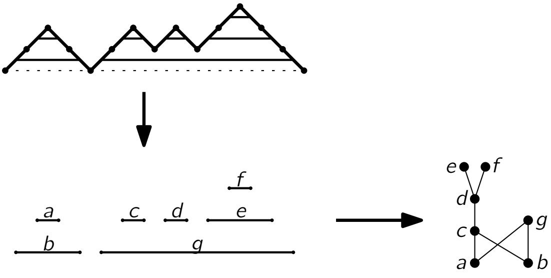

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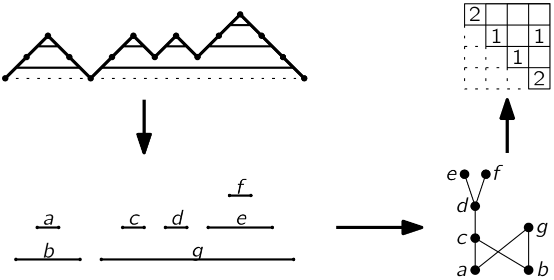

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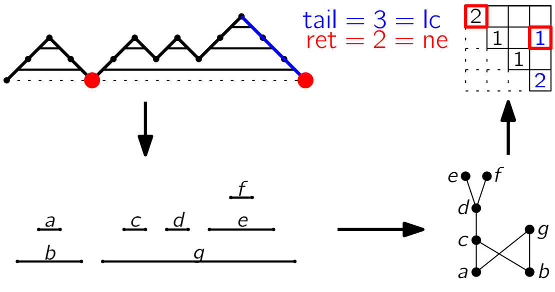

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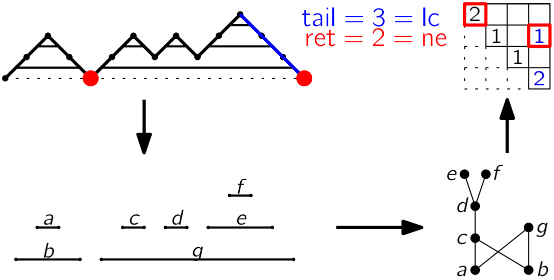

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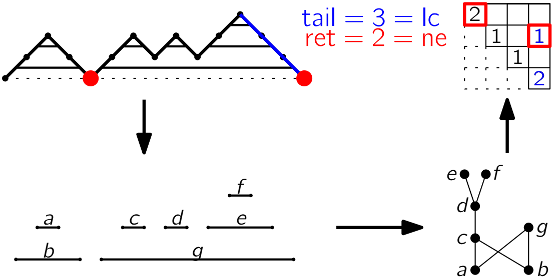

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Dyck Paths ∼ ($\mathbf{2+2}$,$\mathbf{3+1}$)-free Posets ∼ NE-free Matrices

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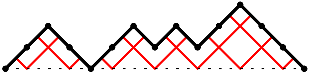
Dyck Paths ∼ (**2**+**2**,**3**+**1**)-free Posets ∼ NE-free Matrices



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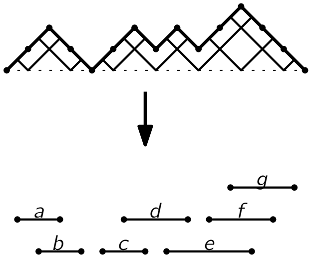

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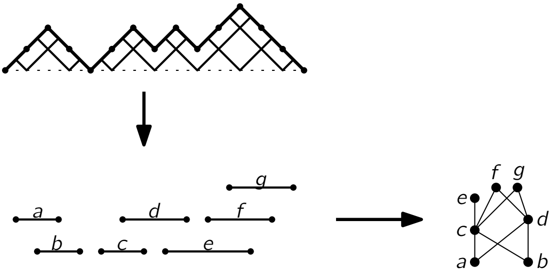

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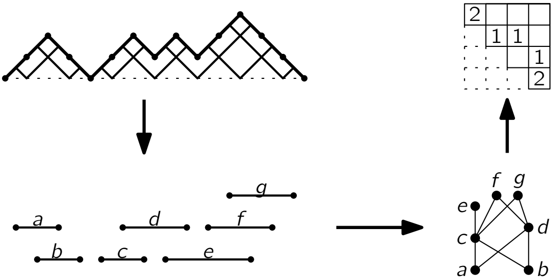

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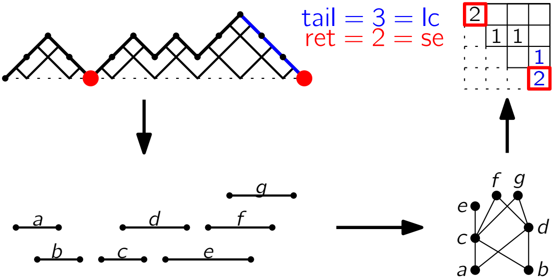

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In particular, se(·) and lc(·) have symmetric joint distribution on NE-free Fishburn matrices.

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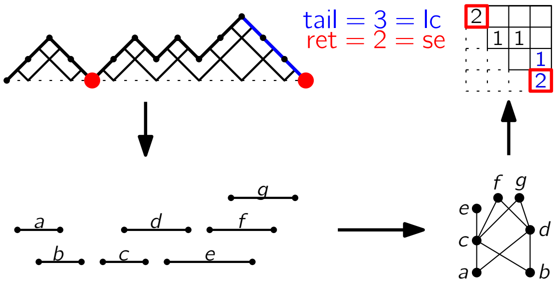

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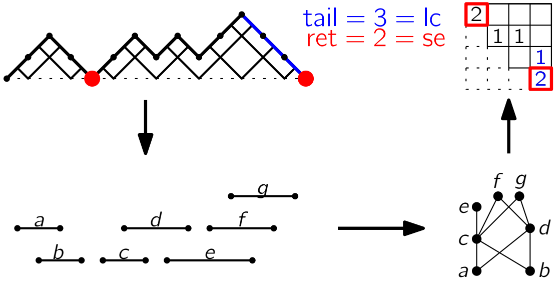

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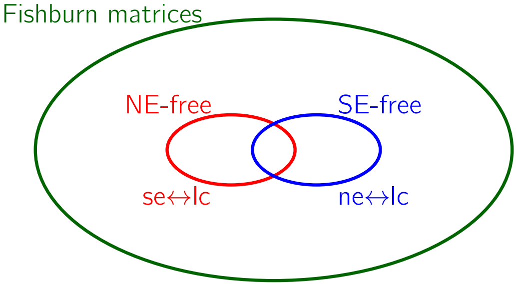
Summary



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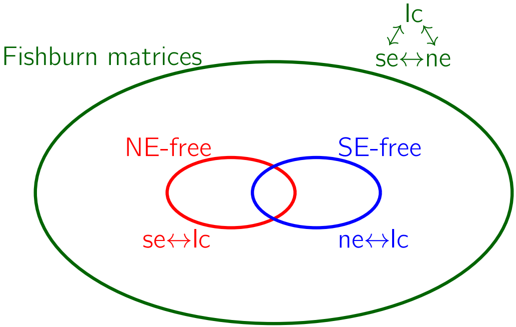

Symmetries of Fishburn Objects

Theorem (J., 2015)

For every n, the two statistics $\mathrm{ne}(\cdot)$ *and* $\mathrm{se}(\cdot)$ *have symmetric joint distribution on Fishburn matrices of weight n. This is witnessed by an involution that preserves the value of* $\mathrm{lc}(\cdot)$.

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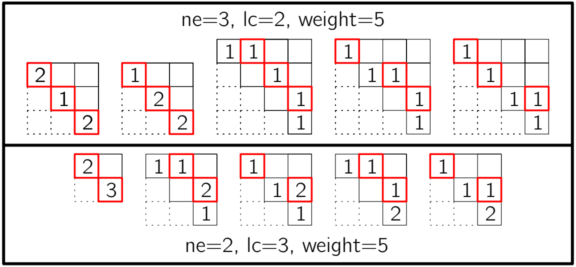

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Open problem

Prove the second theorem bijectively.

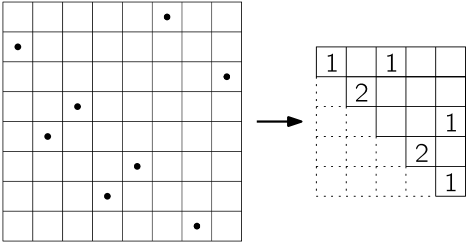
Open Problem: Avoiders of

Conjecture

There is a bijection between Fishburn matrices and permutations avoiding , which transforms the statistics as follows:

LR-maxima	RL-minima	RL-maxima	LR-minima
weight od first row	lc(·)	ne(·)	# of nonzero diagonal cells

Moreover, inverse permutation is mapped to the ‘diagonal flip’ of the matrix.



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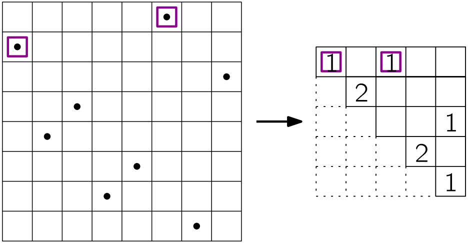

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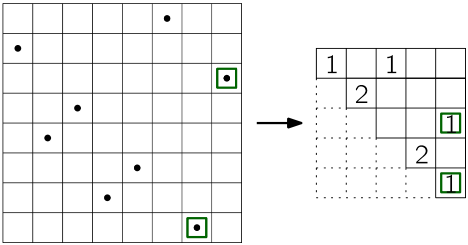

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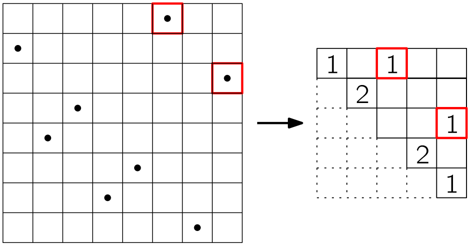

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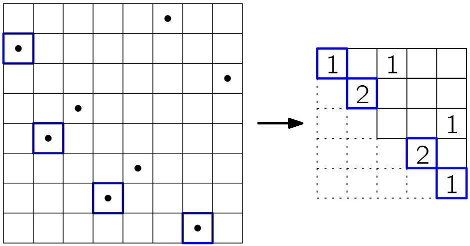

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Conjecture

The statistics "number of RL-maxima" and "number of RL-minima" have symmetric joint distribution over -avoiders of a given size.

Open Problem: Typical Properties of Fishburn Objects

Open problem

What does a uniformly random interval order (or Fishburn matrix) of large size look like? Can you generate it efficiently?

Theorem (Drmota, 2011; Brightwell–Keller, 2011)

The probability that a random Fishburn matrix of weight n is primitive tends to $e^{-\pi^2/6} \sim 0.193$ as $n \to \infty$.

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The End

Thank you for your attention!

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