Interval minors of binary matrices

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Joint work with Vít Jelínek.

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From permutations to binary matrices

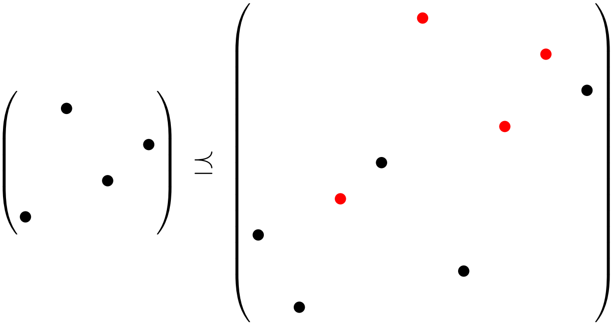
1423

314592687

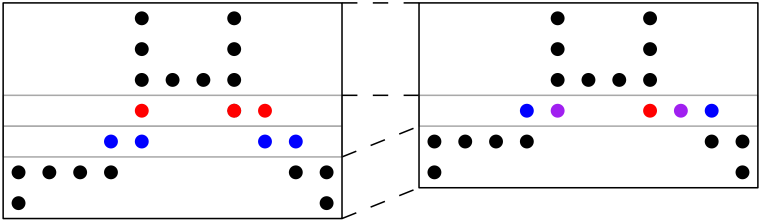
$$\begin{pmatrix} & \bullet & & \\ & & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix} \quad \begin{pmatrix} & & & & \bullet & & & & \\ & & & & & & & \bullet & \\ & & & & & & & & \bullet \\ & & & & & & \bullet & & \\ & & & \bullet & & & & & \\ & & \bullet & & & & & & \\ \bullet & & & & & & & & \\ & & & & & \bullet & & & \\ & \bullet & & & & & & & \end{pmatrix}$$

From permutations to binary matrices

$1423 \quad \preceq \quad 314592687$



Row contraction



Interval minor

Definition 1

A binary matrix P is an interval minor of a binary matrix M, if P can be obtained from M by a sequence of:

- line (row or column) contractions, and
- one-entry deletions.

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Definition 2

A binary matrix $P \in \{0, 1\}^{k \times l}$ is an interval minor of a binary matrix M, if there is a partitioning of M by $k - 1$ horizontal and $l - 1$ vertical lines to $k \cdot l$ rectangular submatrices $M[i, j]$, $1 \leq i \leq k$ and $1 \leq j \leq l$ such that if $P_{i,j}$ is a one-entry then $M[i, j]$ contains a one-entry.

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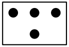
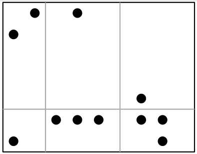
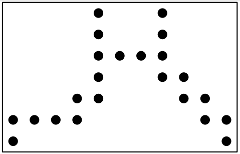
$P =$  $M_1 =$  $M_2 =$ 

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Definition

A binary matrix M is P-critical for a binary matrix P, if P is not an interval minor of M and a change of an arbitrary zero-entry to a one-entry creates a matrix containing an interval minor P.

What do P-critical matrices M look like?

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- We only consider matrices P without empty lines.

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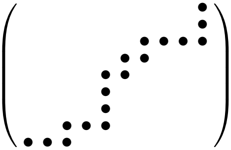
- P with two one-entries:

$$P = \begin{pmatrix} \bullet \\ \bullet \end{pmatrix} \qquad M = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

$$P = \begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix} \qquad M =$$



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- Rotation does not break the property.

What do P-critical matrices M look like?

- P with three one-entries:

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What do they all have in common?
$\rightarrow$ there are at most as many intervals of zero-entries in each row of M as there are columns in P. The same holds for columns of M and rows of P.

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$\rightarrow$ there are at most as many intervals of zero-entries in each row of M as there are columns in P. The same holds for columns of M and rows of P.

Question 1

Is this true for every matrix P?

Zero-intervals and matrix complexity

Definition

A zero-interval in a matrix M is an interval (contiguous sequence) of zero-entries in a row or column of M, bounded from each side by a one-entry or the edge of the matrix.

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$$M = \begin{array}{cc} \begin{pmatrix}
 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
 & & \bullet & & \bullet & \bullet & & & & \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & & &
\end{pmatrix} & \begin{matrix} \\ \\ \\ \\ \leftarrow \\ \\ \\ \\ \\ \end{matrix} \end{array}$$

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 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
\circ & \circ & \bullet & & \bullet & \bullet & & & & \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & & &
\end{pmatrix} \leftarrow$$

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 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
 & & \bullet & \circ & \bullet & \bullet & & & & \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ \leftarrow \\ \\ \\ \\ \\ \end{matrix}$$

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 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
 & & \bullet & & \bullet & \bullet & \circ & \circ & \circ & \circ \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & & &
\end{pmatrix} \leftarrow$$

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 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
\circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ & \circ & \circ \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
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$$M = \begin{pmatrix}
 & & & & & \color{red}{\circ} & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & \color{red}{\circ} & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
\circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ & \circ & \circ \\
 & \bullet & \bullet & & \bullet & \color{red}{\circ} & & & & \\
 & \bullet & & & \bullet & \color{red}{\circ} & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \color{red}{\circ} & & & & \\
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\end{pmatrix}$$

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Definition

The row complexity of a matrix M is the maximum number of zero-intervals in a row of M. The column complexity of a matrix M is the maximum number of zero-intervals in a column of M. The complexity of a matrix M is the maximum of its row and column complexity.

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 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & & & \bullet & \bullet & & & \\
 & & \bullet & & \bullet & \bullet & & & & \\
 & \bullet & \bullet & & \bullet & & & & & \\
 & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & & &
\end{pmatrix}$$

Bounding matrices

Definition

A binary matrix P is bounding if there exists $k \in \mathbb{N}$ such that the complexity of every P-critical matrix is less than or equal to k. Otherwise, P is non-bounding.

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Is every matrix P bounding?

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No!

Bounding matrices – theorem

Theorem 1

A binary matrix P is non-bounding if and only if any rotation of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$ is an interval minor of P.

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Proof: only $\Leftarrow$

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Proof: only $\Leftarrow$

$$P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \qquad \begin{pmatrix} \bullet & & \bullet & & \bullet & \cdots & \bullet & & \bullet & & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \end{pmatrix} \qquad B = \begin{pmatrix} \bullet & & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

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$$P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$$

$$\begin{pmatrix}
\bullet & \bullet & & \bullet & \bullet & \cdots & \bullet & \bullet & & \bullet & \bullet & \bullet & & \bullet & \bullet \\
 & & & & & \cdots & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \cdots & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \cdots & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \cdots & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \cdots & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & & & & & & & & &
\end{pmatrix}$$

$$B = \begin{pmatrix}
\bullet & \bullet & & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
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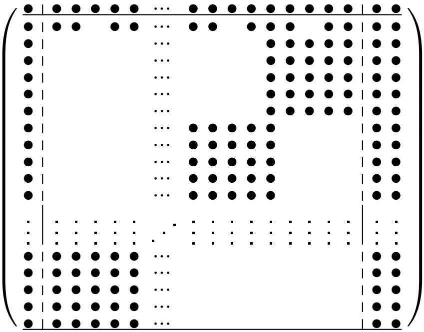

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Corollary 1

If a matrix P is bounding then every P' interval minor of P is bounding.

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Definition

A binary matrix P is row-bounding if there exists $k \in \mathbb{N}$ such that the row complexity of every P-critical matrix is less than or equal to k.

Bounding matrices – corollaries

Theorem 1

A binary matrix P is non-bounding if and only if any rotation of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$ is an interval minor of P.

Corollary 1

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Corollary 2

A matrix P is row-bounding if and only if P is column-bounding.

Difficult one-entries

$$P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \qquad M = \begin{pmatrix} \bullet & & \bullet & & \bullet & \cdots & \bullet & & \bullet & & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \end{pmatrix}$$

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$$P_1 = \begin{pmatrix} & {\color{red}\bullet} & \\ \bullet & & \\ & & \bullet \end{pmatrix} \qquad M = \begin{pmatrix} \bullet & {\color{red}\circ} & \bullet & {\color{red}\circ} & \bullet & \cdots & \bullet & {\color{red}\circ} & \bullet & {\color{red}\circ} & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & & & \bullet & \bullet & \bullet \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ & & & & & \cdots & \bullet & \bullet & \bullet & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ & & \bullet & \bullet & \bullet & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \\ \bullet & \bullet & \bullet & & & \cdots & & & & & \end{pmatrix}$$

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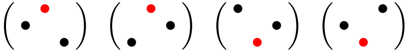
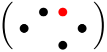
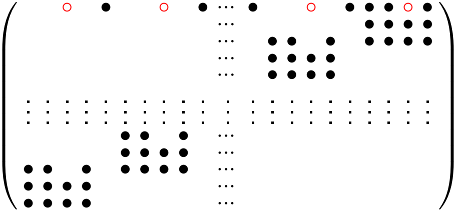
Difficult one-entries



$P =$  $M =$ 

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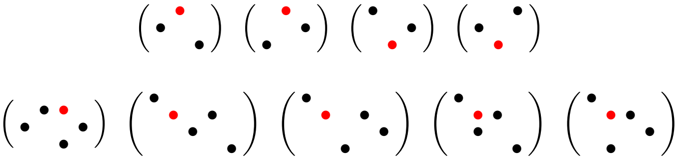

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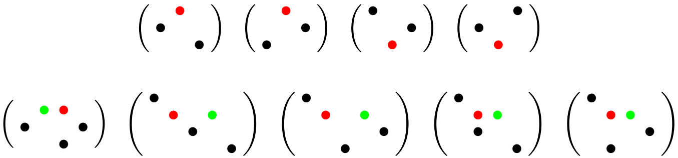

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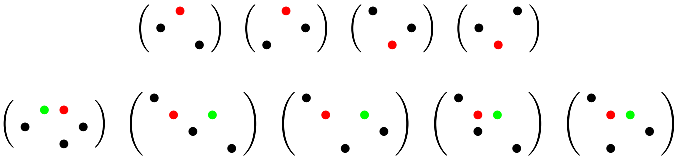

Question

Does every difficult one-entry share the row with a trivially difficult one-entry?

Open problems

- Can Corollary 1 or Corollary 2 be proved without using Theorem 1?
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- What is the magnitude of $\mathrm{f}(P)$ for a bounding binary matrix P?

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- What is the magnitude of $f(P)$ for a bounding binary matrix P? Is it linear? Quadratic? ...
- Is $f(P)$ linear for P of size $2 \times n$?
- Let P be an interval minor of a bounding binary matrix P'. Is $f(P) \leq f(P')$?

Thank you.