Positional Marked Patterns in Permutations

Permutation Patterns 2017

Sittipong Thamrongpairoj, UCSD
Joint work with Jeff Remmel

June 30, 2017

Outline

1. Motivation

Outline

1. Motivation
2. Definition

Outline

1. Motivation
2. Definition
3. Results for patterns of length 3

Outline

1. Motivation
2. Definition
3. Results for patterns of length 3
4. Results for patterns of length 4

Outline

1. Motivation
2. Definition
3. Results for patterns of length 3
4. Results for patterns of length 4
5. Results for patterns of arbitrary length

Outline

1. Motivation
2. Definition
3. Results for patterns of length 3
4. Results for patterns of length 4
5. Results for patterns of arbitrary length
6. Future research

Motivation

Motivation

Remmel and Kitaev studied a notion of marked mesh pattern.

Motivation

Remmel and Kitaev studied a notion of marked mesh pattern.

Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we will consider the graph of σ, $G(\sigma)$ to be the set of points (i, σ_i) for $i = 1, \ldots, n$.

Motivation

Remmel and Kitaev studied a notion of marked mesh pattern.

Given a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we will consider the graph of σ, $G(\sigma)$ to be the set of points (i, σ_i) for $i = 1, \dots, n$.

If we draw a coordinate system centered at a point (i, σ_i), we are interested in the points that lie in the four quadrants I,II,III, and IV of that coordinate system.

Motivation

Given $a, b, c, d \in \mathbb{Z}_{\geq 0}$, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are

Motivation

Given $a, b, c, d \in \mathbb{Z}_{\geq 0}$, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are

- at least a points in quadrant I
- at least b points in quadrant II
- at least c points in quadrant III
- at least d points in quadrant IV

Motivation

Given $a, b, c, d \in \mathbb{Z}_{\geq 0}$, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are

- at least a points in quadrant I
- at least b points in quadrant II
- at least c points in quadrant III
- at least d points in quadrant IV

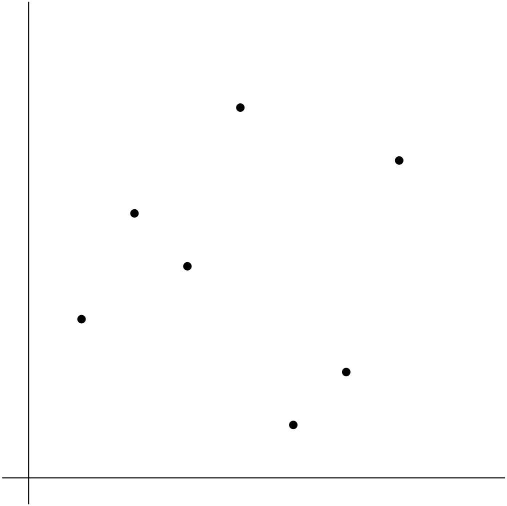
$$\begin{array}{c|c} \geq b & \geq a \\ \hline \geq c & \geq d \end{array}$$

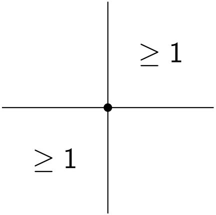
Motivation

Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

Motivation

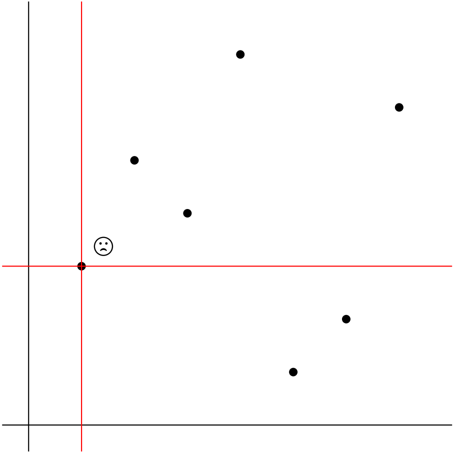
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

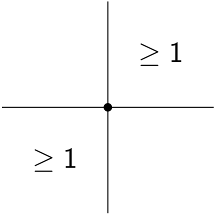




Motivation

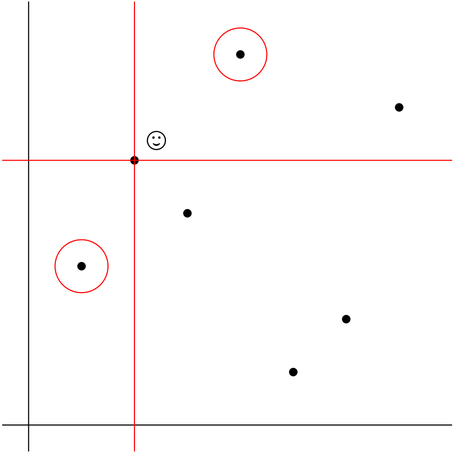
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

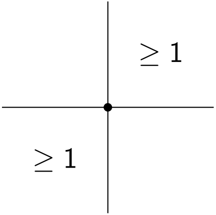




Motivation

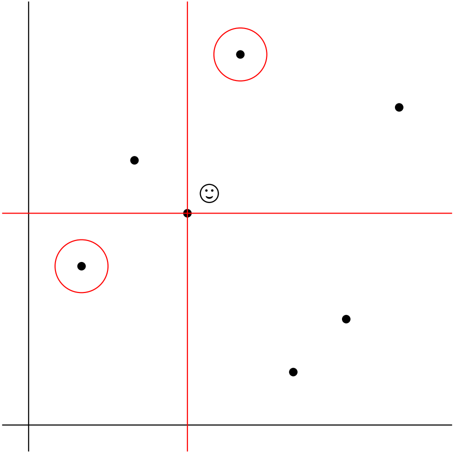
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

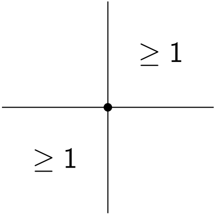




Motivation

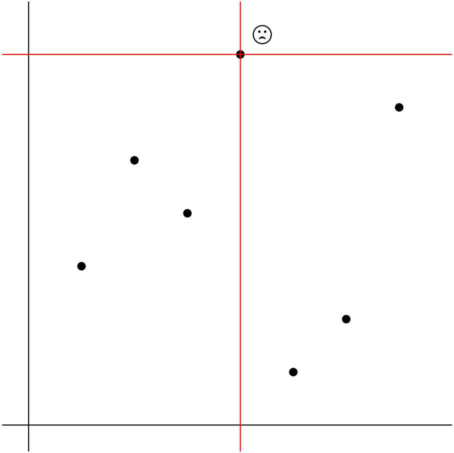
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

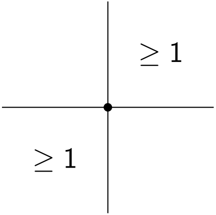




Motivation

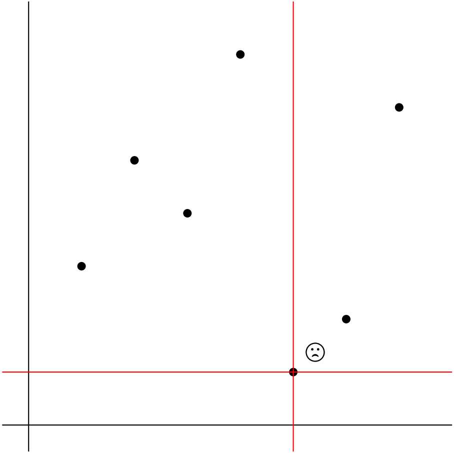
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

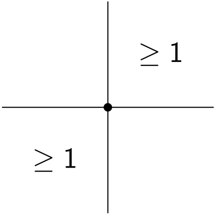




Motivation

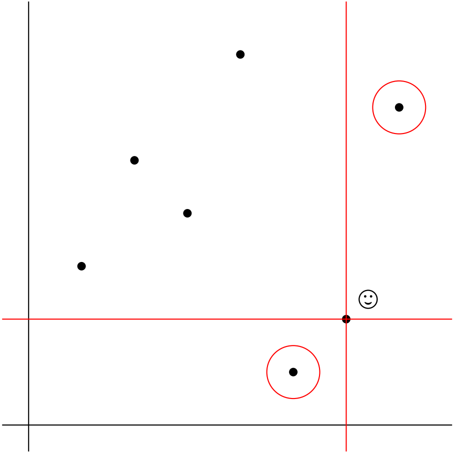
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

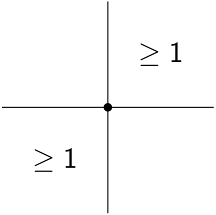




Motivation

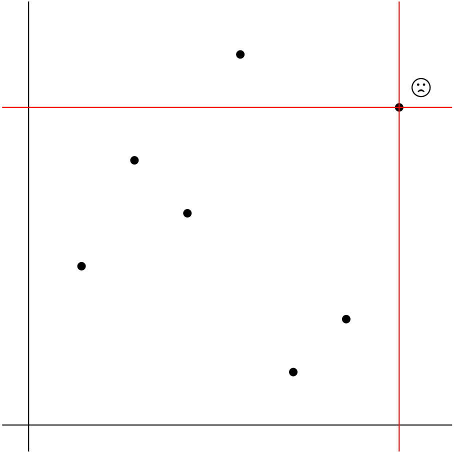
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like

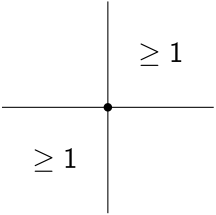




Motivation

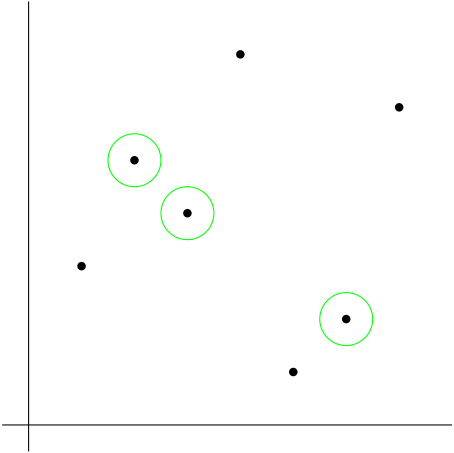
Consider $a = c = 1$ and $b = d = 0$, and $\sigma = 3547126$, then $G(\sigma)$ looks like





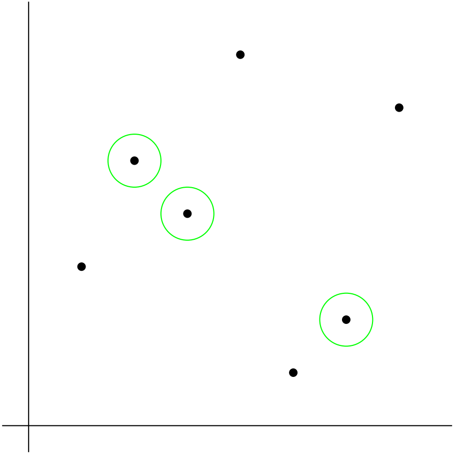
Motivation

We let $mmp_{(a,b,c,d)}(\sigma)$ be the number of i such that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$. Then, we have $mmp_{(1,0,1,0)}(3547126) = 3$



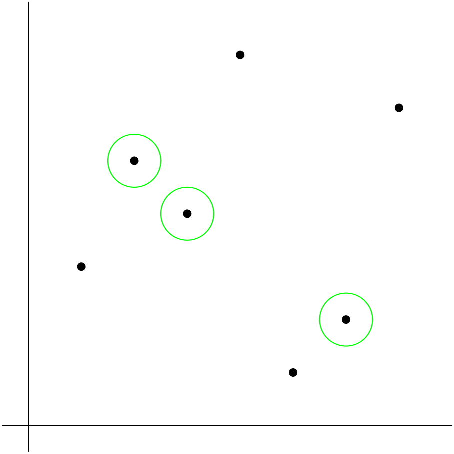
Motivation

Another way to understand $MMP(1,0,1,0)$: **The number of possible midpoint of pattern** 123**.**



Motivation

Another way to understand $MMP(1,0,1,0)$: **The number of possible midpoint of pattern** 123**.**



Why don't we do the same thing for other patterns and other postions?

Definition

Definition

A *positional marked pattern* is pair (τ, s) such that $\tau = \tau_1 \ldots \tau_j \in S_j$ and $1 \leq s \leq j$.

Definition

A *positional marked pattern* is pair (τ, s) such that $\tau = \tau_1 \dots \tau_j \in S_j$ and $1 \leq s \leq j$.

Given a positional marked pattern (τ, s) and a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we say that σ_i matches (τ, s) if there exists $1 \leq i_1 < \cdots < i_{s-1} < i < i_{s+1} < \ldots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \dots \sigma_{i_{s-1}} \sigma_i \sigma_{i_{s+1}} \dots \sigma_{i_j}) = \tau$.

Definition

A *positional marked pattern* is pair (τ, s) such that $\tau = \tau_1 \dots \tau_j \in S_j$ and $1 \leq s \leq j$.

Given a positional marked pattern (τ, s) and a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we say that σ_i matches (τ, s) if there exists $1 \leq i_1 < \dots < i_{s-1} < i < i_{s+1} < \ldots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \dots \sigma_{i_{s-1}} \sigma_i \sigma_{i_{s+1}} \dots \sigma_{i_j}) = \tau$.

In the language of classical patterns, σ_i **matches** (τ, s) **if there is an occurrence of** τ **in** σ **such that** σ_i **plays the role of** τ_s **in that occurrence.**

Definition

A *positional marked pattern* is pair (τ, s) such that $\tau = \tau_1 \ldots \tau_j \in S_j$ and $1 \leq s \leq j$.

Given a positional marked pattern (τ, s) and a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches (τ, s) if there exists $1 \leq i_1 < \cdots < i_{s-1} < i < i_{s+1} < \ldots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_{s-1}} \sigma_i \sigma_{i_{s+1}} \ldots \sigma_{i_j}) = \tau$.

In the language of classical patterns, σ_i **matches** (τ, s) **if there is an occurrence of** τ **in** σ **such that** σ_i **plays the role of** τ_s **in that occurrence.**

We let $pmp_{(\tau,s)}(\sigma)$ denote the number of i such that σ_i matches (τ, s) in σ

Definition

As an example, consider $\sigma = 632541$ and pattern $(321, 1)$.

Definition

As an example, consider $\sigma = 632541$ and pattern $(321, 1)$.There are 8 occurrences of pattern 321 in σ, namely,

Definition

As an example, consider $\sigma = 632541$ and pattern $(321, 1)$.There are 8 occurrences of pattern 321 in σ, namely,

$$632, 631, 621, 654, 651, 641, 321, 541$$

Definition

As an example, consider $\sigma = 632541$ and pattern $(321, 1)$.There are 8 occurrences of pattern 321 in σ, namely,

$$632, 631, 621, 654, 651, 641, 321, 541$$

To compute $pmp_{(321,1)}(\sigma)$, we shall count the number of elements that can be the starting point of pattern 321.

Definition

As an example, consider $\sigma = 632541$ and pattern $(321, 1)$.There are 8 occurrences of pattern 321 in σ, namely,

$$632, 631, 621, 654, 651, 641, 321, 541$$

To compute $pmp_{(321,1)}(\sigma)$, we shall count the number of elements that can be the starting point of pattern 321.

We see that 6,3 and 5 can play the role of 3 in 321, so $pmp_{(321,1)}(\sigma) = 3$.

Definition

Back to marked mesh pattern, $MMP(1,0,1,0)$ is the same as $(123,2)$.

Goal

Given a positive integer n and a positional marked pattern (τ, s), we are interested in enumerating polynomials of the form

$$P_{n,(\tau,s)}(x) = \sum_{\sigma \in S_n} x^{pmp_{n,(\tau,s)}}$$

Goal

Given a positive integer n and a positional marked pattern (τ, s), we are interested in enumerating polynomials of the form

$$P_{n,(\tau,s)}(x) = \sum_{\sigma \in S_n} x^{pmp_{n,(\tau,s)}}$$

Given two positional marked patterns $(\alpha, s), (\beta, t)$, we say that (α, s) is positional Wilf-equivalent to (β, t) if

$$P_{n,(\alpha,s)}(x) = P_{n,(\beta,t)}(x)$$

Goal

Given a positive integer n and a positional marked pattern (τ, s), we are interested in enumerating polynomials of the form

$$P_{n,(\tau,s)}(x) = \sum_{\sigma \in S_n} x^{pmp_{n,(\tau,s)}}$$

Given two positional marked patterns $(\alpha, s), (\beta, t)$, we say that (α, s) is positional Wilf-equivalent to (β, t) if

$$P_{n,(\alpha,s)}(x) = P_{n,(\beta,t)}(x)$$

Goal: Classify equivalent classes for positional marked patterns

Connection to classical patterns

Given any positional marked pattern (τ, s), $P_{n,(\tau,s)}(0)$ is just the number of permutations σ that avoid τ.

Connection to classical patterns

Given any positional marked pattern (τ, s), $P_{n,(\tau,s)}(0)$ is just the number of permutations σ that avoid τ.

Thus, if (α, s) and (β, t) are positional Wilf-equivalent, then α and β are Wilf-equivalent.

Connection to classical patterns

Given any positional marked pattern (τ, s), $P_{n,(\tau,s)}(0)$ is just the number of permutations σ that avoid τ.

Thus, if (α, s) and (β, t) are positional Wilf-equivalent, then α and β are Wilf-equivalent.

One can think of positional marked pattern as a refinement of classical pattern.

Alternate representation of positional marked patterns

For small patterns, we shall often denote (τ, s) by underlining the τ_s in τ.

Alternate representation of positional marked patterns

For small patterns, we shall often denote (τ, s) by underlining the τ_s in τ.

For example, we denote $(1432, 2)$ by $1\underline{4}32$

Alternate representation of positional marked patterns

For small patterns, we shall often denote (τ, s) by underlining the τ_s in τ.

For example, we denote $(1432, 2)$ by $1\underline{4}32$

Morevoer, we associate a positional marked pattern with a permutation matrix-like diagram. Use $\circ$ for underlined element, and $\times$ for other elements. For example, $1\underline{4}32$ corresponds to the diagram

Alternate representation of positional marked patterns

For small patterns, we shall often denote (τ, s) by underlining the τ_s in τ.

For example, we denote $(1432, 2)$ by $1\underline{4}32$

Morevoer, we associate a positional marked pattern with a permutation matrix-like diagram. Use $\circ$ for underlined element, and $\times$ for other elements. For example, $1\underline{4}32$ corresponds to the diagram

	$\circ$		
		$\times$	
			$\times$
$\times$			

Patterns of length 3

Patterns of length 3

It is easy to see that two positional marked patterns are equivalent if the diagram of one can be transformed into the other via a series of rotations and reflection.

Patterns of length 3

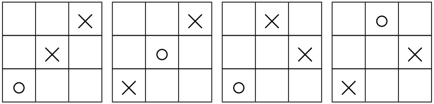
It is easy to see that two positional marked patterns are equivalent if the diagram of one can be transformed into the other via a series of rotations and reflection.

Consider the positional marked patterns of length 3. There are $3! \cdot 3 = 18$ positional marked patterns of length 3. However, with symmetry, there are at most 4 equivalence classes, which are represented by $\underline{1}23, 1\underline{2}3, \underline{1}32, 1\underline{3}2$.

Patterns of length 3

It is easy to see that two positional marked patterns are equivalent if the diagram of one can be transformed into the other via a series of rotations and reflection.

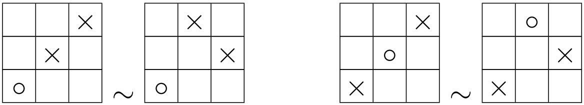
Consider the positional marked patterns of length 3. There are $3! \cdot 3 = 18$ positional marked patterns of length 3. However, with symmetry, there are at most 4 equivalence classes, which are represented by $\underline{1}23, 1\underline{2}3, \underline{1}32, 1\underline{3}2$. Their diagrams are:



Patterns of length 3

Theorem (Remmel, T)

$\underline{1}23$ and $\underline{1}32$ are positional Wilf-equivalent. $1\underline{2}3$ and $1\underline{3}2$ are positional Wilf-equivalent. However, $\underline{1}23$ is not positional Wilf-equivalent to $1\underline{2}3$.



Sketch of a proof that $\underline{1}23$ and $\underline{1}32$
Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. So σ has a form

Patterns of length 3

Sketch of a proof that $\underline{1}23$ and $\underline{1}32$

Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. So σ has a form

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k < \sigma_{k+1} > \sigma_{k+2} > \ldots > \sigma_n$$

Sketch of a proof that $\underline{1}23$ **and** $\underline{1}32$
Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. So σ has a form

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k < \sigma_{k+1} > \sigma_{k+2} > \ldots > \sigma_n$$

Considering the position of 1 in σ, we have

Sketch of a proof that $\underline{1}23$ **and** $\underline{1}32$

Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. So σ has a form

$$\sigma = \sigma_1 \ \sigma_2 \ldots \sigma_{k-1} \ \sigma_k < \sigma_{k+1} > \sigma_{k+2} > \ldots > \sigma_n$$

Considering the position of 1 in σ, we have

$$\begin{array}{rcll} P_{n,\tau_1,k}(x) & = & (k-1)xP_{n-1,\tau_1,k-1}(x) & \text{if 1 is at position before } k \\ & & +1 + \sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x) & \text{if 1 is at position } k \\ & & +P_{n-1,\tau_1,k}(x) & \text{if 1 is at position } n \end{array}$$

Sketch of a proof that $\underline{1}23$ **and** $\underline{1}32$

Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. So σ has a form

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k < \sigma_{k+1} > \sigma_{k+2} > \ldots > \sigma_n$$

Considering the position of 1 in σ, we have

$$\begin{array}{rcll} P_{n,\tau_1,k}(x) & = & (k-1)xP_{n-1,\tau_1,k-1}(x) & \text{if 1 is at position before } k \\ & & +1 + \sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x) & \text{if 1 is at position } k \\ & & +P_{n-1,\tau_1,k}(x) & \text{if 1 is at position } n \end{array}$$

$$= (k-1)xP_{n-1,\tau_1,k-1}(x) + 1 + \sum_{l=1}^{k} P_{n-1,\tau_1,l}(x)$$

Patterns of length 3

Do the same for $\underline{1}32$

Patterns of length 3

Do the same for $\underline{1}32$
Let $\tau_2 = \underline{1}32$ and let $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last descent of σ is at position k. So σ has a form

Patterns of length 3

Do the same for $\underline{1}32$

Let $\tau_2 = \underline{1}32$ and let $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last descent of σ is at position k. So σ has a form

$$\sigma = \sigma_1 \ \sigma_2 \ldots \sigma_{k-1} \ \sigma_k > \sigma_{k+1} < \sigma_{k+2} < \ldots < \sigma_n$$

Patterns of length 3

Do the same for $\underline{1}32$
Let $\tau_2 = \underline{1}32$ and let $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last descent of σ is at position k. So σ has a form

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k > \sigma_{k+1} < \sigma_{k+2} < \ldots < \sigma_n$$

Considering the position of 1 in σ, we have

Patterns of length 3

Do the same for $\underline{1}32$
Let $\tau_2 = \underline{1}32$ and let $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last descent of σ is at position k. So σ has a form

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k > \sigma_{k+1} < \sigma_{k+2} < \ldots < \sigma_n$$

Considering the position of 1 in σ, we have

$$\begin{array}{rcll} P_{n,\tau_2,k}(x) & = & (k-1)xP_{n-1,\tau_2,k-1}(x) & \text{if 1 is at position before } k \\ & & +1 + \sum_{l=1}^{k} P_{n-1,\tau_2,l}(x) & \text{if 1 is at position } k+1 \end{array}$$

Patterns of length 3

$P_{n,\tau_1,k}(x)$ and $P_{n,\tau_2,k}(x)$ have the same recursive formula. One can check that they have the same initial conditions. Hence, τ_1 and τ_2 are positional Wilf-equivalent. $\square$

Patterns of length 3

Enumeration of $P_{n,\underline{1}23}(x) = P_{n,\underline{1}32}(x)$

$P_{1,\underline{1}23}(x) = 1$

$P_{2,\underline{1}23}(x) = 2$

$P_{3,\underline{1}23}(x) = 5 + x$

$P_{4,\underline{1}23}(x) = 14 + 8x + 2x^2$

$P_{5,\underline{1}23}(x) = 42 + 47x + 25x^2 + 6x^3$

$P_{6,\underline{1}23}(x) = 132 + 244x + 216x^2 + 104x^3 + 24x^4$

$P_{7,\underline{1}23}(x) = 429 + 1186x + 1568x^2 + 1199x^3 + 538x^4 + 120x^5$

Patterns of length 3

Enumeration of $P_{n,\underline{1}23}(x) = P_{n,\underline{1}32}(x)$

$$P_{1,\underline{1}23}(x) = 1$$
$$P_{2,\underline{1}23}(x) = 2$$
$$P_{3,\underline{1}23}(x) = 5 + x$$
$$P_{4,\underline{1}23}(x) = 14 + 8x + 2x^2$$
$$P_{5,\underline{1}23}(x) = 42 + 47x + 25x^2 + 6x^3$$
$$P_{6,\underline{1}23}(x) = 132 + 244x + 216x^2 + 104x^3 + 24x^4$$
$$P_{7,\underline{1}23}(x) = 429 + 1186x + 1568x^2 + 1199x^3 + 538x^4 + 120x^5$$

Catalan numbers

Patterns of length 3

Enumeration of $P_{n,\underline{1}23}(x) = P_{n,\underline{1}32}(x)$

$$P_{1,\underline{1}23}(x) = 1$$

$$P_{2,\underline{1}23}(x) = 2$$

$$P_{3,\underline{1}23}(x) = 5 + x$$

$$P_{4,\underline{1}23}(x) = 14 + 8x + 2x^2$$

$$P_{5,\underline{1}23}(x) = 42 + 47x + 25x^2 + 6x^3$$

$$P_{6,\underline{1}23}(x) = 132 + 244x + 216x^2 + 104x^3 + 24x^4$$

$$P_{7,\underline{1}23}(x) = 429 + 1186x + 1568x^2 + 1199x^3 + 538x^4 + 120x^5$$

A139262

Patterns of length 3

Enumeration of $P_{n,\underline{1}23}(x) = P_{n,\underline{1}32}(x)$

$P_{1,\underline{1}23}(x) = 1$

$P_{2,\underline{1}23}(x) = 2$

$P_{3,\underline{1}23}(x) = 5 + x$

$P_{4,\underline{1}23}(x) = 14 + 8x + 2x^2$

$P_{5,\underline{1}23}(x) = 42 + 47x + 25x^2 + 6x^3$

$P_{6,\underline{1}23}(x) = 132 + 244x + 216x^2 + 104x^3 + 24x^4$

$P_{7,\underline{1}23}(x) = 429 + 1186x + 1568x^2 + 1199x^3 + 538x^4 + 120x^5$

$(n-2)!$

Patterns of length 3

Theorem (Remmel, T)

$P_{n,\underline{1}32}(x)|_x = \sum_{\sigma \in S_n(132)} inv(\sigma)$

Theorem (Remmel, T)

Given any positive integers n, k such that $n \geq k$, and any $\tau \in S_k$ and $1 \leq s \leq k$, the degree of $P_{n,(\tau,s)}(x)$ is $n-k+1$, and $P_{n,(\tau,s)}(x)|_{x^{n-k+1}} = (n-k+1)!$.

Patterns of length 3

To prove that $\begin{array}{|c|c|c|}\hline & & \times \\ \hline & \circ & \\ \hline \times & & \\ \hline \end{array} \sim \begin{array}{|c|c|c|}\hline & \circ & \\ \hline & & \times \\ \hline \times & & \\ \hline \end{array}$, we keep track of the position of 1 and derive a recursive formula based on the position of 2.

Patterns of length 3

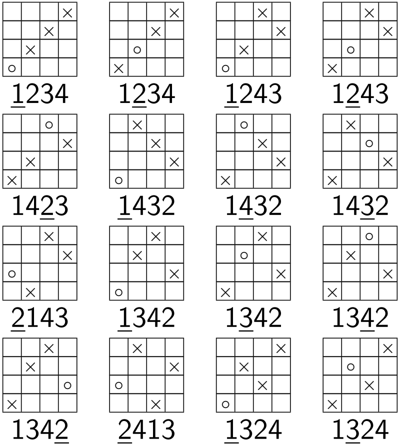
Enumeration of $P_{n,1\underline{2}3}(x) = P_{n,1\underline{3}2}(x)$

$$\begin{aligned}
P_{1,1\underline{2}3}(x) &= 1 \\
P_{2,1\underline{2}3}(x) &= 2 \\
P_{3,1\underline{2}3}(x) &= 5 + x \\
P_{4,1\underline{2}3}(x) &= 14 + 8x + 2x^2 \\
P_{5,1\underline{2}3}(x) &= 42 + 46x + 26x^2 + 6x^3 \\
P_{6,1\underline{2}3}(x) &= 132 + 232x + 220x^2 + 112x^3 + 24x^4 \\
P_{7,1\underline{2}3}(x) &= 429 + 1093x + 1527x^2 + 1275x^3 + 596x^4 + 120x^5
\end{aligned}$$

Patterns of length 4

Patterns of length 4

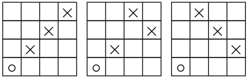
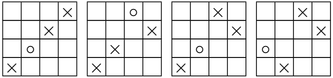
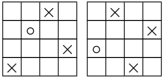
By symmetry, there are at most 16 positional Wilf-equivalent classes:

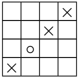
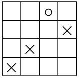
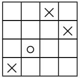
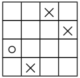


Patterns of length 4

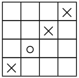
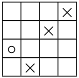
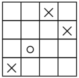
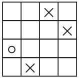
Numerical data suggests that there are 10 classes.

Numerical data suggests that there are 10 classes.

1. 
   $\underline{1}234$ $\underline{1}243$ $\underline{1}432$
2. 
   $1\underline{2}34$ $14\underline{2}3$ $1\underline{2}43$ $\underline{2}143$
3. 
   $1\underline{3}42$ $\underline{2}413$
4. Everything else forms its own class

Consider the class $1\underline{2}34$ $14\underline{2}3$ $1\underline{2}43$ $\underline{2}143$

Equivalently, we can look at $1\underline{2}34$ $\underline{2}134$ $1\underline{2}43$ $\underline{2}143$

Theorem (Remmel, T)

Four patterns $1\underline{2}34, \underline{2}134, 1\underline{2}43, \underline{2}143$ are positional Wilf-equivalent.

Patterns of length 4

Idea of the proof:
$1\underline{2}34$ and $1\underline{2}43$, refine the polynomials by last ascent/descent and position of 1. Derive recursive formula by consider the position of 2.

$1\underline{2}34$ and $\underline{2}134$ as well as $1\underline{2}43$ and $\underline{2}143$ follow from a general theorem.

Patterns of arbitrary length

Patterns of arbitrary length

Theorem (Remmel, T)

Let $p_1, p_2, \ldots, p_k$ be a rearrangement of $3, 4, \ldots, k+2$, then $1\underline{2}p_1 p_2 \ldots p_k$ and $\underline{2}1p_1 p_2 \ldots p_k$. are positional Wilf-equivalent.

Patterns of arbitrary length

Theorem (Remmel, T)

Let $p_1, p_2, \ldots, p_k$ be a rearrangement of $3, 4, \ldots, k+2$, then $1\underline{2}p_1p_2\ldots p_k$ and $\underline{2}1p_1p_2\ldots p_k$. are positional Wilf-equivalent.

With this theorem, we deduce that $1\underline{2}34$ and $\underline{2}134$ are positional Wilf-equivalent, as well as $1\underline{2}43$ and $\underline{2}143$ are equivalent.

Patterns of arbitrary length

Sketch of the proof: As an example, we will use $1\underline{2}34$ and $\underline{2}134$.

Patterns of arbitrary length

Sketch of the proof: As an example, we will use $1\underline{2}34$ and $\underline{2}134$.

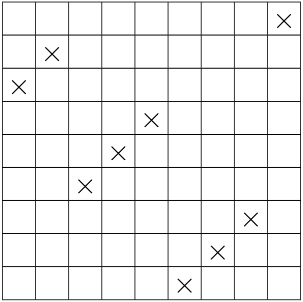
Step 1: Associates each permutations with their "non-dominant part."

Patterns of arbitrary length

Sketch of the proof: As an example, we will use $1\underline{2}34$ and $\underline{2}134$.

Step 1: Associates each permutations with their “non-dominant part.”

Given any σ, find all $\times$ in the diagram of σ such that there is no copy of pattern 12 to the top right of it.

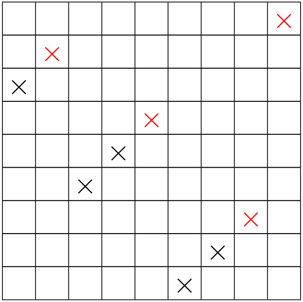


Patterns of arbitrary length

Sketch of the proof: As an example, we will use $1\underline{2}34$ and $\underline{2}134$.

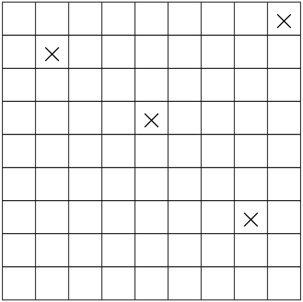
Step 1: Associates each permutations with their "non-dominant part."

Given any σ, find all $\times$ in the diagram of σ such that there is no copy of pattern 12 to the top right of it.



Fix a non-dominant part, consider all σ that have non-dominant part as given.

Key 1: All such σ can be obtained by starting with a non-dominant part and filling the rest of the diagram:



Patterns of arbitrary length

Fix a non-dominant part, consider all σ that have non-dominant part as given.

Key 1: All such σ can be obtained by starting with a non-dominant part and filling the rest of the diagram:

Put $\bullet$ in "non-dominant" cell.

$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\times$
$\bullet$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\bullet$	$\bullet$	$\bullet$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
				$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
				$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
				$\bullet$	$\bullet$	$\bullet$	$\times$	$\bullet$
							$\bullet$	$\bullet$
							$\bullet$	$\bullet$

Patterns of arbitrary length

Fix a non-dominant part, consider all σ that have non-dominant part as given.

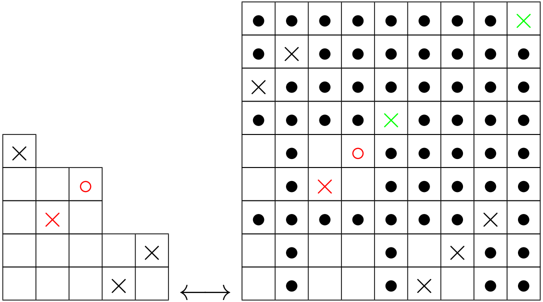
Key 1: All such σ can be obtained by starting with a non-dominant part and filling the rest of the diagram:

Put $\bullet$ in rows and columns that already contain $\times$.

$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\times$
$\bullet$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\bullet$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\bullet$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\times$	$\bullet$
	$\bullet$			$\bullet$			$\bullet$	$\bullet$
	$\bullet$			$\bullet$			$\bullet$	$\bullet$

Patterns of arbitrary length

Key 2: $1\underline{2}\tau$-matching in σ corresponds to $1\underline{2}$-matching in the smaller part.



Patterns of arbitrary length

Key 3: $1\underline{2}$ and $\underline{2}1$ are equivalent for smaller boards.

Patterns of arbitrary length

Key 3: $1\underline{2}$ and $\underline{2}1$ are equivalent for smaller boards.

Conclusion: We can refine the set of permutations by their non-dominant part, and prove the generating functions for two patterns are the same considering each refinement. So, the whole generating functions are the same.

Future research

Future research

Future research

1. Equivalence of $\underline{1}234$, $\underline{1}432$ and $\underline{1}243$

Future research

1. Equivalence of $\underline{1}234$, $\underline{1}432$ and $\underline{1}243$
2. Equivalence of $12\underline{3}\tau$ and $\underline{3}21\tau$

Future research

1. Equivalence of $\underline{1}234$, $\underline{1}432$ and $\underline{1}243$
2. Equivalence of $12\underline{3}\tau$ and $\underline{3}21\tau$
3. Positional marked pattern on words

Future research

1. Equivalence of $\underline{1}234$, $\underline{1}432$ and $\underline{1}243$
2. Equivalence of $12\underline{3}\tau$ and $\underline{3}21\tau$
3. Positional marked pattern on words
4. Multiple positional marked patterns: σ_i matches $\{\tau_1, \tau_2\}$ if σ_i matches τ_1 or τ_2.

References

P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, Electronic J. Combin. **18** (2) (2011), Paper #P5.

S. Kitaev and J. Remmel, Quadrant marked mesh patterns, Journal of Integer Sequence, **15** Issue 4 (2012), Article 12.4.7, 29 pgs.

Henning A. Úlfarsson, A unification of permutation patterns related to Schubert varieties, DMTCS proc., **AN** (2010), 1057-1068.

E. Babson and J. West, The Permutations 123p4 . . . pm and 321p4 . . . pm are Wilf-Equivalent, Graphs and Combinatorics, **16:173**, (2000), 373-380

Thank you for you attention!

Thank you for you attention!

Thank you for you attention!

Thank you for you attention!