Inversion Sequences and Generating Trees

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Permutation Patterns 2017

Inversion Sequences

- An *inversion sequence* is an integer sequence $e_1 \dots e_n$ satisfying $0 \leq e_i < i$ for all $i = 1, \dots, n$.
- Inversion sequences are naturally bijective to permutations:
  $e = \Theta(\pi)$ is obtained from a permutation $\pi = \pi_1 \dots \pi_n$ by setting $e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|$.
- The study of patterns in inversion sequences was introduced in:
  - inversion sequences avoiding permutations of length 3 [T. Mansour, M. Shattuck 2015].
  - inversion sequences that avoid words of length 3 [S. Corteel, M. A. Martinez, C. D. Savage, M. Weselcouch 2016]
- An inversion sequence avoids a pattern $a_1\, a_2\, a_3$ if there are not three indices $i < j < k$ such that $e_i\, e_j\, e_k = a_1\, a_2\, a_3$.

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Inversion Sequences

Example

- $I_n(110)$: sequences with no $i < j < k$ such that $e_i = e_j > e_k$.
- corresponds to the permutation $\pi = 9\,6\,10\,3\,8\,4\,1\,7\,5\,2$.

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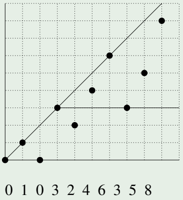

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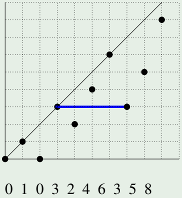

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Inversion Sequences Avoiding Triples of Relations

- Martinez and Savage generalized the notion of pattern avoidance to a triple of binary relations (ρ_1, ρ_2, ρ_3), where $\rho_i \in \{<, >, \leq, \geq, =, \neq, -\}$, where $-$ on a set S is the cartesian product, i.e. $- = S \times S$.
- $I_n(\rho_1, \rho_2, \rho_3)$ is the set of inversion sequences e of length n with no $i < j < k$ such that

$$e_i \rho_1 e_j, \quad e_j \rho_2 e_k, \quad e_i \rho_3 e_k.$$

- For example $I_n(=, >, >) = I_n(110)$.
- All triples of relations of the set $\{<, >, \leq, \geq, =, \neq, -\}^3$ are studied in [Martinez, Savage 2016].
- All 343 patterns are considered and partitioned in 98 equivalence classes. Several conjectures are formulated.

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Inversion Sequences Avoiding Patterns of Length 3

Inversion sequences with no $i<j<k$ such that:	appear to be counted by OEIS seq:	proven?	notes/OEIS description	a_7, equiv class	Section
$e_i \neq e_j$ and $e_i \neq e_k$	A004275	yes	$2(n-1)$ for $n>1$	12,A	2.2
$e_i \geq e_j$ and $e_i \neq e_k$	A004275	yes	$2(n-1)$ for $n>1$	12,B	2.2
$e_i = e_j \leq e_k$	A000045	yes	Fibonacci numbers, F_{n+1}	21	2.3
$e_i < e_j \neq e_k$	A000124	yes	Lazy caterer sequence	22,A	2.4
$e_i < e_j$ and $e_i < e_k$	A000124	yes	Lazy caterer sequence	22,B	2.4
$e_i \geq e_j \neq e_k$	A000124	yes	Lazy caterer sequence	22,C	2.4
$e_i \neq e_j \leq e_k$	A000071	yes	$F_{n+2}-1$	33,A	2.5
$e_i \geq e_j \leq e_k$ and $e_i \neq e_k$	A000071	yes	$F_{n+2}-1$	33,B	2.5
$e_i = e_j < e_k$	A000079	yes	$\mathbf{I}_n(001)$, 2^{n-1} (see [13])	64,A	2.6
$e_i < e_j \leq e_k$	A000079	yes	2^{n-1}	64,B	2.6
$e_i < e_j \geq e_k$	A000079	yes	2^{n-1}	64,C	2.6
$e_i \leq e_j = e_k$	A000079	yes	2^{n-1}	64,D	2.6
$e_i \neq e_j < e_k$	A000325	yes	Grassmannian permutations	121,A	2.7
$e_i \neq e_j \neq e_k$ and $e_i \neq e_k$	A000325	yes	Grassmannian permutations	121,B	2.7
$e_j \geq e_k$ and $e_i \neq e_k$	A000325	yes	Grassmannian permutations	121,C	2.7
$e_i \neq e_j < e_k$ and $e_i \leq e_k$	A034943	yes	321-avoiding separable perms	151	2.8
$e_i \neq e_j < e_k$ and $e_i \neq e_k$	A088921	yes	$\mathbf{S}_n(321, 2143)$	185	2.9
$e_i \geq e_k$	A049125	no	ordered trees, internal nodes adj. to ≤ 1 leaf	187	2.10
$e_i \leq e_j \geq e_k$ and $e_i \neq e_k$	A005183	yes	$\mathbf{S}_n(132, 4312)$, $n2^{n-1}+1$	193	2.11
$e_i < e_j < e_k$	A001519	yes	$\mathbf{I}_n(012)$, F_{2n-1} (see [13, 21])	233	2.12
$e_i = e_k$	A229046	no	recurrence $\rightarrow$ gf?	304	2.13
$e_j > e_k$	A000108	yes	Catalan numbers	429,A	2.14
$e_j \geq e_k$ and $e_i < e_k$	A000108	yes	Catalan numbers	429,B	2.14
$e_i \geq e_j$ and $e_i \geq e_k$	A000108	no	Catalan numbers	429,C	2.14
$e_i \neq e_j = e_k$	A047970	yes	$\mathbf{S}_n(\bar{3}\bar{1}542)$, nexus numbers	523	2.15
$e_j \leq e_k$ and $e_i \geq e_k$	A108307	no	set partitions avoiding enhanced 3-crossings	772,A	2.16
$e_i \geq e_j \geq e_k$	A108307	no	set partitions avoiding enhanced 3-crossings	772,B	2.16
$e_i < e_j = e_k$	A000110	yes	$\mathbf{I}_n(011)$ (see [13]), Bell numbers B_n	877,A	2.17
$e_i = e_j \geq e_k$	A000110	no	$\mathbf{I}_n(000, 110)$, B_n	877,B	2.17
$e_j \neq e_k$ and $e_i = e_k$	A000110	yes	$\mathbf{I}_n(010, 101)$, B_n	877,C	2.17
$e_i \geq e_j$ and $e_i = e_k$	A000110	no	$\mathbf{I}_n(000, 101)$, B_n	877,D	2.17
$e_i > e_j$	A000984	yes	central binomial coefficients	924	2.18
$e_i > e_j \leq e_k$	A071356	no	certain underdiagonal lattice paths	1064	2.19
$e_i > e_j < e_k$	A033321	yes	$\mathbf{S}_n(2143, 3142, 4132)$ (see [8])	1265	2.20
$e_i > e_j$ and $e_i \leq e_k$	A106228	no	$\mathbf{I}_N(101, 102)$, $\mathbf{S}_n(4123, 4132, 4213)$	1347	2.21
$e_i = e_j = e_k$	A000111	yes	$\mathbf{I}_n(000)$ (see [13]), Euler up/down numbers	1385	2.22
$e_i > e_j$ and $e_i < e_k$	A200753	yes	$\mathbf{I}_n(102)$, [21]	1694	2.23
$e_j > e_k$ and $e_i < e_k$	A006318	yes	$\mathbf{I}_n(021)$ [13, 21], large Schröder numbers R_{n-1}	1806,A	2.24
$e_i > e_j$ and $e_i \geq e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 101, 100)$, R_{n-1}	1806,B	2.24
$e_i \geq e_j$ and $e_i > e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 100, 110)$, R_{n-1}	1806,C	2.24
$e_i \geq e_j \neq e_k$ and $e_i \geq e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 101, 110)$, R_{n-1}	1806,D	2.24
$e_i \geq e_j \geq e_k$ and $e_i > e_k$	A001181	no	Baxter permutations	2074	2.25
$e_i > e_j$ and $e_i > e_k$	A098746	no	$\mathbf{I}_n(210, 201, 100)$, $\mathbf{S}_n(4231, 42513)$	2549,A	2.26
$e_i > e_j \neq e_k$ and $e_i \geq e_k$	A098746	no	$\mathbf{I}_n(210, 201, 101)$, $\mathbf{S}_n(4231, 42513)$	2549,B	2.26
$e_i \geq e_j \neq e_k$ and $e_i > e_k$	A098746	no	$\mathbf{I}_n(210, 201, 110)$, $\mathbf{S}_n(4231, 42513)$	2549,C	2.26
$e_j < e_k$ and $e_i \geq e_k$	A117106	no	$\mathbf{I}_n(201, 101)$, $\mathbf{S}_n(21\bar{3}54)$	2958,A	2.27
$e_i > e_j \geq e_k$	A117106	no	$\mathbf{I}_n(210, 100)$, $\mathbf{S}_n(21\bar{3}54)$	2958,B	2.27
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$e_j < e_k$ and $e_i = e_k$	A113227	yes	$\mathbf{I}_n(101)$, $\mathbf{S}_n$(1-23-4), (see [13])	3207,A	2.28
$e_i = e_j > e_k$	A113227	yes	$\mathbf{I}_n(110)$, $\mathbf{S}_n$(1-23-4), (see [13])	3207,B	2.28
$e_i > e_j \neq e_k$ and $e_i > e_k$	A212198	yes	$\mathbf{I}_n(201, 210)$, $MMP(0,2,0,2)$-avoiding perms	3720	2.28

Table 2: Patterns whose avoidance sequences appear to match sequences in the OEIS. Those marked as "yes" are cited, if known, and otherwise are proven in this paper.

ECO Method

- ECO method (Enumeration of Combinatorial Objects) was developed by some researchers of the Universities of Florence and Sienna [Barcucci, Del Lungo, Pergola, Pinzani 1999].
- Let $\mathcal{C}$ be a combinatorial class, that is to say any set of discrete objects equipped with a notion of size, such that there is a finite number of objects $\mathcal{C}_n$ of size n for any integer n. Assume also that $\mathcal{C}_1$ contains exactly one object.
- A function $\vartheta : \mathcal{C}_n \rightarrow \mathcal{P}(\mathcal{C}_{n+1})$ is an *ECO operator* if:
  1. for any $O_1, O_2 \in \mathcal{C}_n$, we have $\vartheta(O_1) \cap \vartheta(O_2) = \emptyset$,
  2. for any $O' \in \mathcal{C}_{n+1}$ there is $O \in \mathcal{C}_n$ such that $O' \in \vartheta(O)$.
- Every object of size $n+1$ is uniquely obtained from an object of size n through the application of ϑ.

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Generating trees

- The growth described by ϑ can be represented by means of a generating tree: a rooted infinite tree whose vertices are the objects of $\mathcal{C}$. The objects having the same size lie at the same level (the element of $\mathcal{C}_1$ is at the root), and the sons of an object are the objects it produces through ϑ.
- If the recursive growth described by ϑ is sufficiently regular, then it can be described by means of a succession rule, i.e. a system of the form:

$$\begin{cases} (a) \\ (k) \rightsquigarrow (e_1)(e_2)\dots(e_k). \end{cases},$$

where $(a), (k), (e_i) \in \mathbb{N}^k$.
- Succession rules (or generating trees) have been studied by West (1995) and Banderier, Bousquet-Mélou, Denise, Flajolet, Gardy, Gouyou-Beauchamps (2005).

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Generating trees

- The growth described by ϑ can be represented by means of a generating tree: a rooted infinite tree whose vertices are the objects of $\mathcal{C}$. The objects having the same size lie at the same level (the element of $\mathcal{C}_1$ is at the root), and the sons of an object are the objects it produces through ϑ.
- If the recursive growth described by ϑ is sufficiently regular, then it can be described by means of a succession rule, i.e. a system of the form:

  $$\begin{cases} (a) \\ (k) \rightsquigarrow (e_1)(e_2)\dots(e_k). \end{cases},$$

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A first example

- Non decreasing sequences $I_n(10)$: inversion sequences such that $e_1 = 0$ and $e_{i+1} \geq e_i$.
- Enumerated by Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$.
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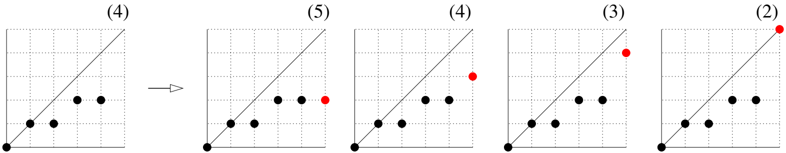

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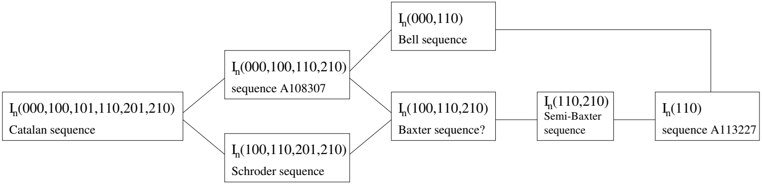

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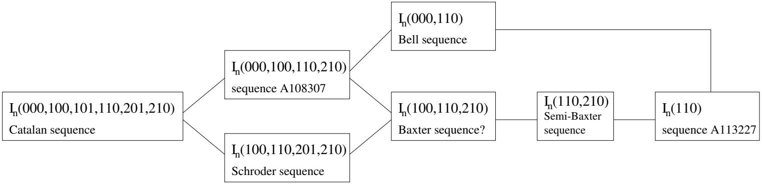

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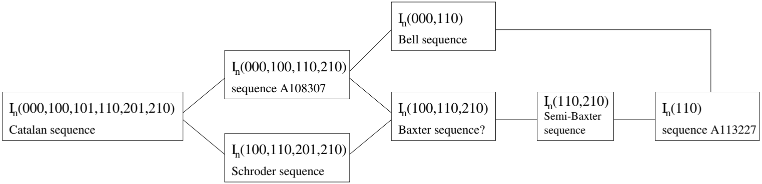

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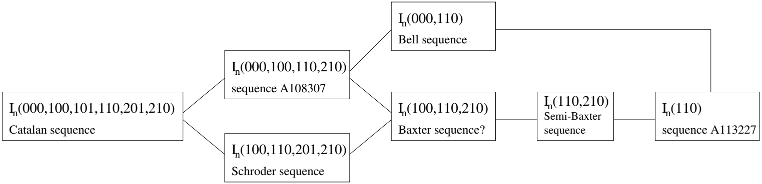

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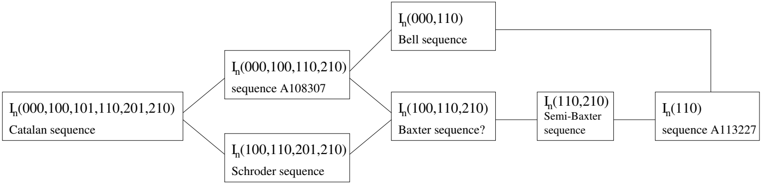

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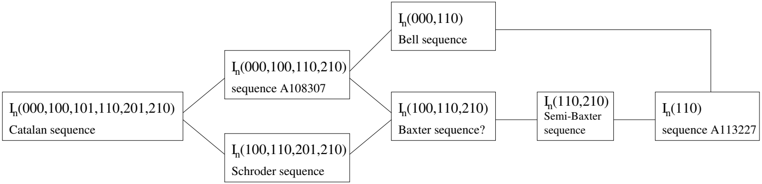

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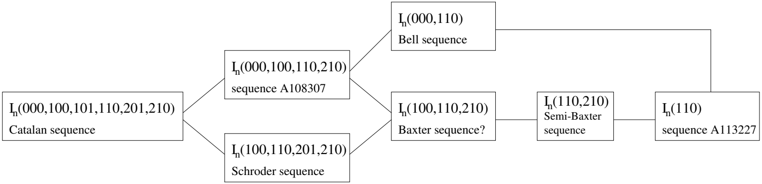

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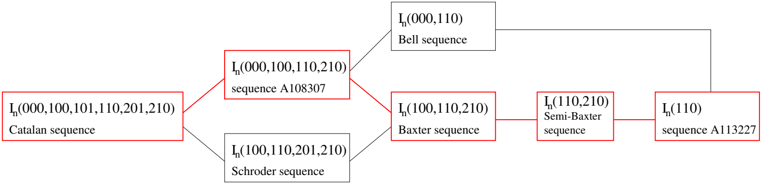

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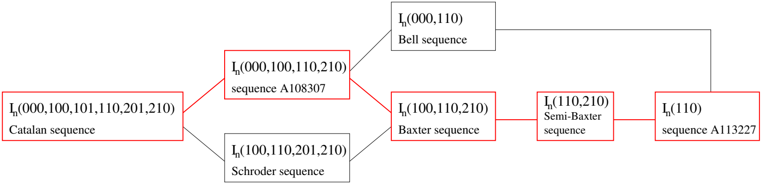

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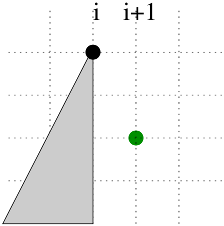

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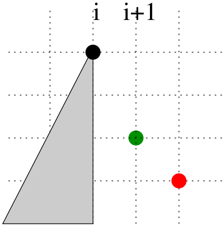

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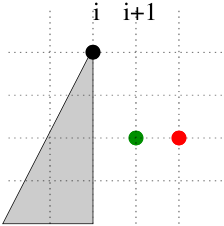

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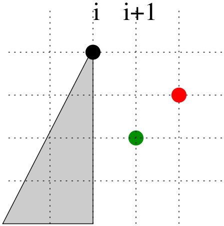

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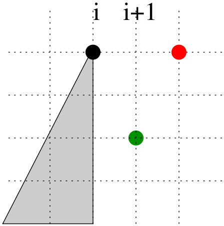

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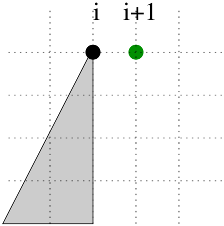

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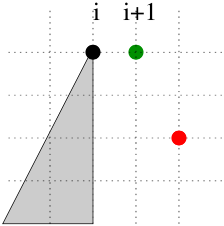

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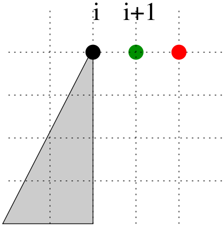

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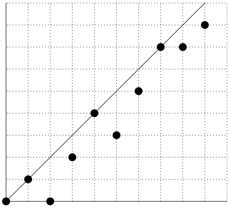

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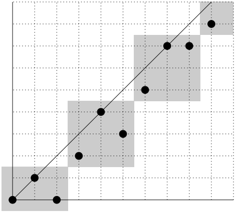

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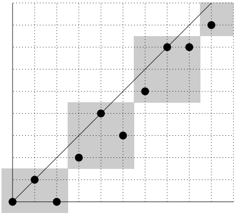

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Catalan sequence: a bijective proof

Proposition

There is a bijective correspondence between sequences of I_n^{cat} and non-crossing partitions of n.

- A partition of $[n] = \{1, \ldots, n\}$ is a pairwise disjoint set of non-empty subsets, called blocks, whose union is $[n]$.
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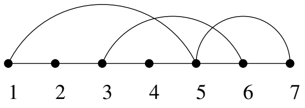

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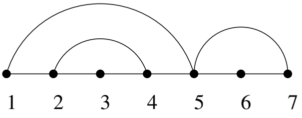

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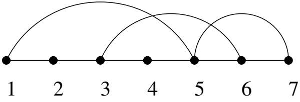

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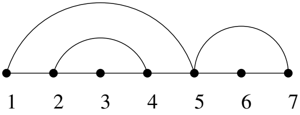

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- We build the associated sequence in the following steps:

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0						
0	1					
0	1	2				
0	1	1	2			
0	0	1	1	2		
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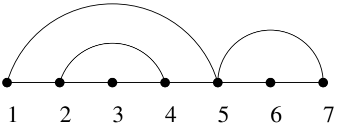

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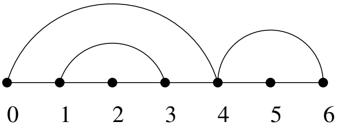

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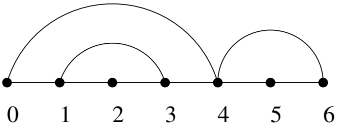

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A more general result

Proposition

The previous construction establishes a bijection between partitions of $[n]$ *(Bell numbers) and* $I_n(000, 110)$.

- Crossing partition:
- Gives:

0	1	2	3	4	5	6	
0							
0	1						
0	1	2					
0	1	2	3				
0	0	1	2	3			
0	0	2	1	2	3		pattern 101
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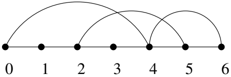
- Crossing partition: 

- Gives:

0	1	2	3	4	5	6	
0							
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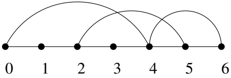
- Crossing partition: 

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A generic ECO operator for inversion sequences

Our general approach

- Let $\mathcal{C}$ be a family of inversion sequences.
- Let a sequence grow by adding an element x at the end of e, and denote by $e \cdot x$ the sequence $e_1 \dots e_n x$.
- An element $x \in \{0, \dots, n\}$ is *active* if $e_1 \dots e_n x \in \mathcal{C}$.
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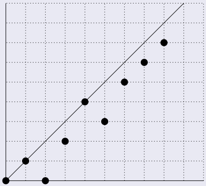

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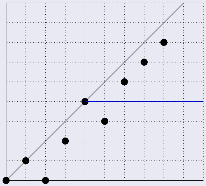

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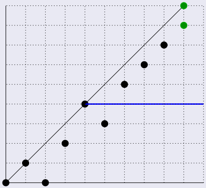

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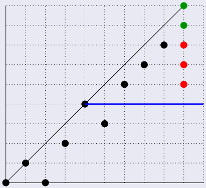

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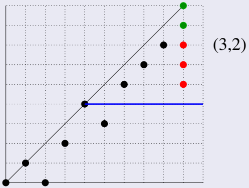

Catalan sequence: a generating tree

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I_n^{cat} grows according to the generating tree:

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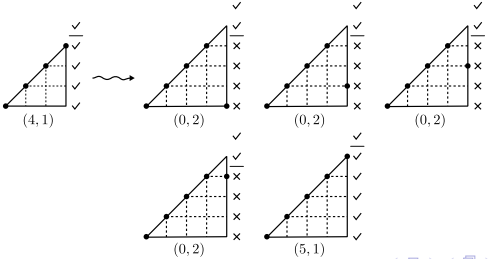

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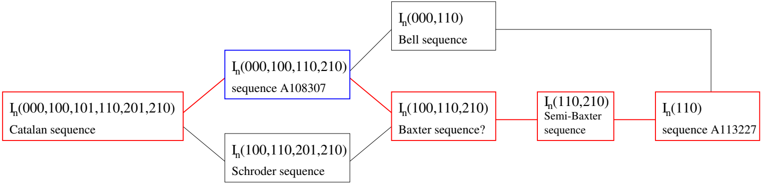

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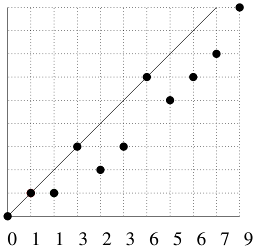

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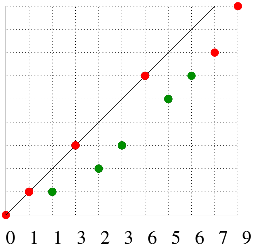

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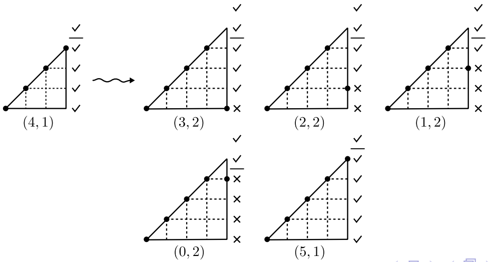

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Let $S_{h,k}(t) \equiv S_{h,k}$ *the gf of* $I_n(000, 100, 110, 210)$ *with label* (h, k)*, and* $S(t; u, v) \equiv S(u, v) = \sum_{h,k \geq 1} S_{h,k} u^h v^k$*. Then:*

$$S(u, v) = tuv + \frac{tv(S(1, v) - S(u, v))}{1 - u} + \frac{tu(S(u, u) - S(u, v))}{u/v - 1}$$

Our "recipe" is as follows:

- Apply some variants of the kernel method (obstinate kernel method) developed in [M. Bousquet-Mélou, G. Xin, 2006] and prove that the gf is D-finite;
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The numbers $b_n = |I_n(000, 100, 110, 210)|$ *satisfy the following polynomial recurrence relation:*

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- Martinez, Savage (2016) conjectured that $\{b_n\}_{n\geq 0}$ is sequence A108307 in The Online Encyclopedia of Integer Sequences.
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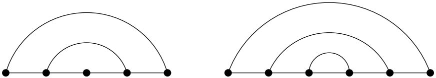

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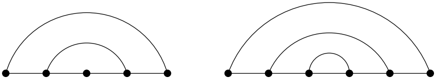

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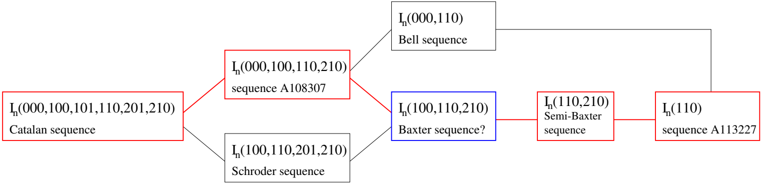

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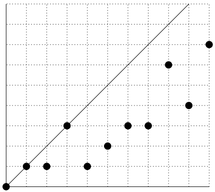

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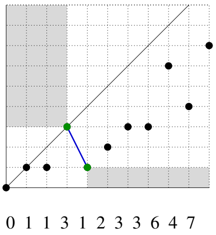

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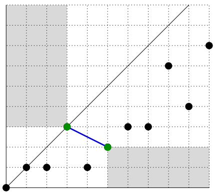

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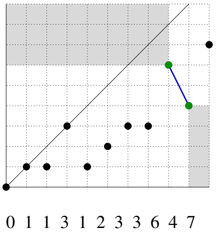

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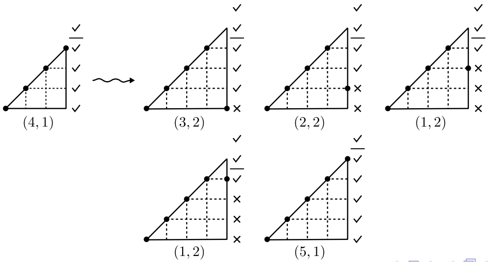

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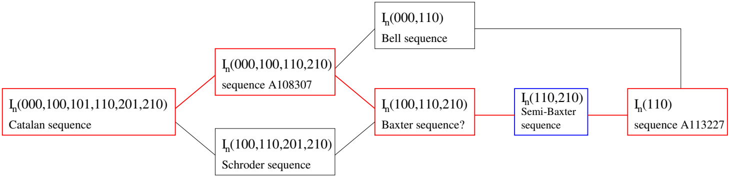

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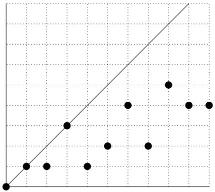

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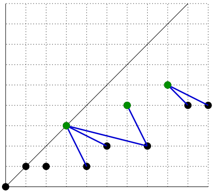

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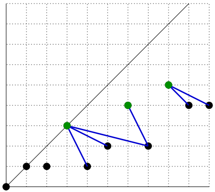

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Semi-Baxter permutations $= AV_n(2\text{-}41\text{-}3)$ (recall that Baxter permutations $= AV_n(2\text{-}41\text{-}3, 3\text{-}14\text{-}2)$).
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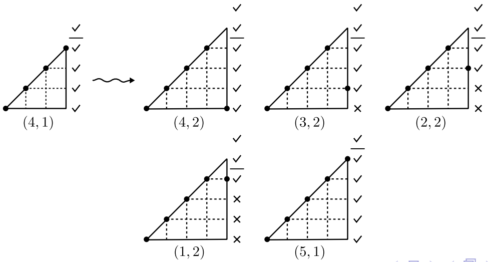

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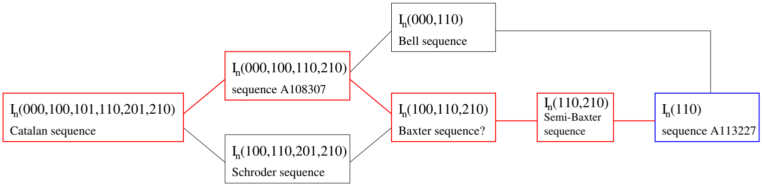

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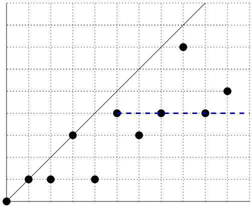

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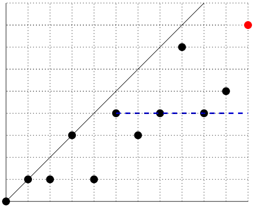

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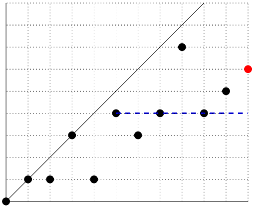

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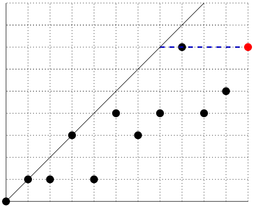

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$$\begin{cases} p_{1,1} = 1, \\ p_{n,j} = p_{n-1,j-1} + j \sum_{i=j}^{n-1} p_{n-1,i} . \end{cases}$$

  Thus, $\{p_n\}_{n \geq 0}$ is sequence A113227 in OEIS.
- Sequence A113227 has been studied by D. Callan (2010), and it is proved to count several families of objects:
  - *marked valleys Dyck paths*,
  - *increasing ordered trees with increasing leaves*,
  - *permutations avoiding* 1-23-4,
  - *steady paths* (equivalent to $AV_n(1\text{-}34\text{-}2)$).

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$I_n(110)$: generating tree

Proposition

$I_n(110)$ *grows according to the generating tree:*

$$\Omega_\ell = \begin{cases} (2) \\ (h) \rightsquigarrow (1)(2)^2 \dots (h)^h (h+1) \end{cases}$$

Proof.

- To a sequence $e \in I_n(110)$ with h occurrences of 0 we assign the label (h).
- The ECO operator applied to e produces objects of size $n+1$ as follows:
  - add 0 on the left of e and increase by 1 all nonzero entries, obtaining a sequence without 1s and label $(h+1)$;

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For every $j = 1, \ldots, h$, the operator produces $h - j + 1$ objects with label $(h - j + 1)$ as follows:

- All entries different from 0 increase by 1;
- The $j - 1$ rightmost entries of 0 become 1;
- One of the $h - j + 1$ remaining entries of 0 becomes 1 (there are $h - j + 1$ possible choices);
- Add 0 at the beginning.

Let $e = 0\,0\,1\,2\,0\,3\,5\,0\,4\,0\,3$ with label (5), and let $j = 2$; we have $5 - 2 + 1 = 4$ productions with label (4):

$$
\begin{array}{rc}
e = & 0\,0\,1\,2\,0\,3\,5\,0\,4\,0\,3 \\
 & \downarrow \\
 & {\color{blue}0}\,{\color{green}1}\,0\,2\,3\,0\,4\,6\,0\,5\,{\color{red}1}\,4 \\
 & {\color{blue}0}\,0\,{\color{green}1}\,2\,3\,0\,4\,6\,0\,5\,{\color{red}1}\,4 \\
 & {\color{blue}0}\,0\,0\,2\,3\,{\color{green}1}\,4\,6\,0\,5\,{\color{red}1}\,4 \\
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$$\Omega_{cat} = \begin{cases} (1) \\ (k) \rightsquigarrow (1)(2) \dots (k)(k+1) \end{cases}$$

  but is not related to the other considered generating trees.
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  - to find a growth of $I_n(110)$ which extends that of $I_n(110, 210)$;
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