Finding functional equations for two 2 by 4 classes

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Permutation Patterns: Reykjavik, Iceland

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Joint work with Jay Pantone

Enumeration of 2 by 4 classes

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Let $\sigma, \tau \in S_4$, and let $\mathcal{C} = Av(\sigma, \tau)$. We call $\mathcal{C}$ a *2 by 4* class.

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Example:

Some 2 by 4 enumeration sequences:

- $Av(1234, 4321) : 1, 2, 6, 22, 86, 306, 882, 1764, \ldots$
- $Av(1234, 1324) : 1, 2, 6, 22, 90, 396, 1837, 8864, \ldots$

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- Albert et al. (2015) argued that three of the remaining classes do not have differentially algebraic generating functions.
- Here, we describe methods for finding functional equations, and generating functions, for the two remaining classes.

The class $Av(3412, 2413)$

Notation:

Let $f(z,t)$ represent the generating function for $Av(3412, 2413)$, where z marks the length of the permutation, and t the length of the initial decreasing sequence.

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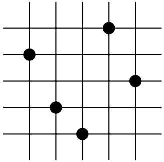

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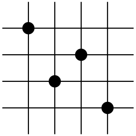

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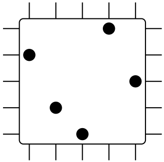

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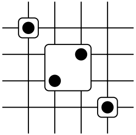

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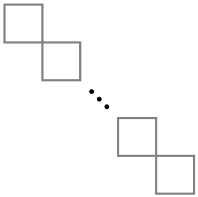

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Figure: The two types of skew decompositions of permutations in $Av(2413, 3412)$. The permutation α is skew-indecomposable of length at least two.

This observation yields the equation

$$f = \frac{1}{1 - zt} + \frac{f_i}{(1 - zt)(1 - z)}.$$

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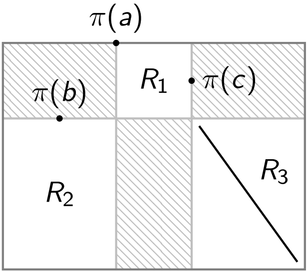

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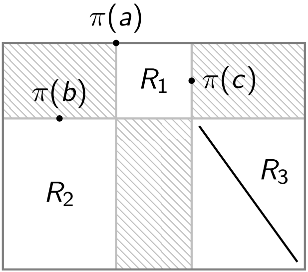

Note that if there are elements in R_3, they must be decreasing due to $\pi(a)$ and $\pi(b)$.

Structure of elements counted by f_i

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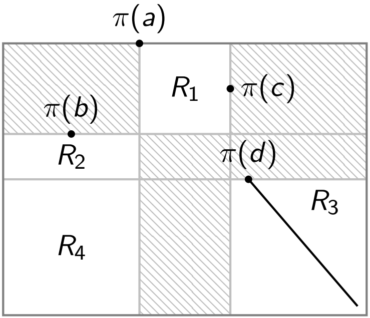

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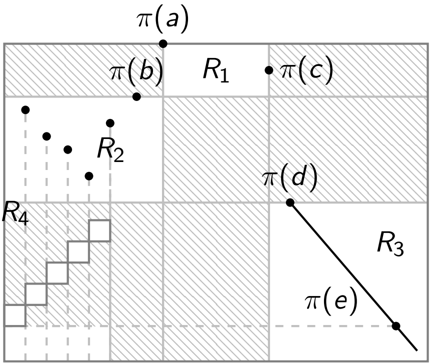

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We can grow our permutation by adding sequences in R_3, followed by elements in R_4, in what we will call *layers*.

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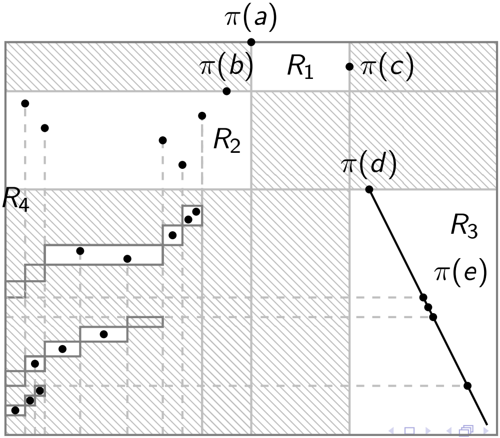

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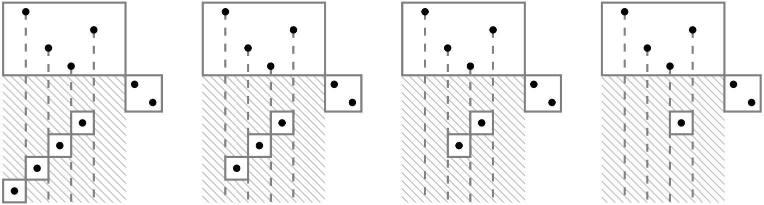

Structure of f_i

Calculations:

Working through the algebra, we get

$$t^3 \mapsto \frac{z}{1-z}(f(z,t)-1)\left[f(z,1)^3 + tf(z,1)^2 + t^2 f(z,1) + t^3\right]$$

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Summary:

Summing over all t, and combining our earlier observations, we have

$$\begin{aligned} f_i(z,t) =& z\,(f(z,t)-1)\,f(z,1) + \frac{z}{1-z}(f(z,t)-1) \\ & \times \left[\frac{f(z,1)f_i(z,f(z,1)) - tf_i(z,t)}{f(z,1)-t}\right]. \end{aligned}$$

Finding the generating function

Using functional equations

Subsequent steps:

- Generate the series for both f and f_i term-by-term
- Guess solutions for f and f_i with the help of Maple
- Verify them and their minimal polynomials using algebra and Gröbner bases

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Conclusion:

In the end, we find that $f(z,1) = F(z)$ is algebraic of degree 3, with minimal polynomial

$$z^4F(z)^3 + (5z^3 - 11z^2)F(z)^2 + (3z^2 + 10z - 1)F(z) - 9z + 1.$$

Requires around 35 terms to guess. The exponential growth rate of the coefficients of $F(z)$ is $\frac{32}{5}$.

Structure of $Av(2143, 1432)$

New class:

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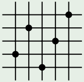
Let $\mathcal{G}(z, t)$ be the generating function for permutations in $Av(2143, 1432)$, where z measures length and t measures the length of the final increasing run (except the first element).

Example:

$\sigma = (2, 4, 1, 3, 5) =$  contributes $z^5 t^3$.

Structure of $\mathcal{V}$

Lemma:

Let $\mathcal{V}(z,t)$ track permutations counted by $\mathcal{G}$ which begin with their smallest element. Then the generating function $\mathcal{V}(z,t)$ is given by

$$\mathcal{V}(z,t) = \frac{z}{1-zt} + \frac{tz^3}{(1-tz)^2(1-2z)} + \frac{t^2z^5}{(1-z)^2(1-tz)^2(1-(1+t)z)}.$$

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The three terms come from permutations which are strictly increasing, those which contain a 21 but not a 2413, and those which do contain a 2413.

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Example

Permutations which contain a 2413 are counted by $\frac{t^2z^5}{(1-z)^2(1-tz)^2(1-(1+t)z)}$, and those with a 21 but not a 2413 are counted by $\frac{tz^3}{(1-tz)^2(1-2z)}$.

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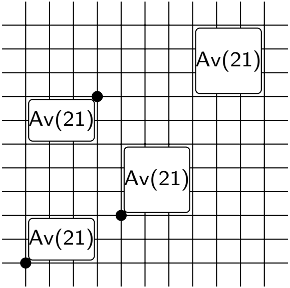

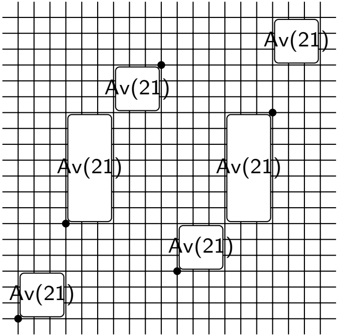

Functional equation for $Av(2143, 1432)$

Based on the structure of permutations in the class, we find that the generating function $\mathcal{G}$ satisfies the equation

$$\mathcal{G}(z,t) = 1 + \mathcal{V}\left(\frac{\mathcal{G}(z,1) - tG(z,t)}{1-t}\right)$$
$$+ \frac{t^2z^4}{(1-z)^2(1-tz)(1-(1+t)z)}\left(\frac{\mathcal{G}_t(z,1)}{1-t} - \frac{t(\mathcal{G}(z,1)-\mathcal{G}(z,t))}{(1-t)^2}\right)$$
$$+ \frac{tz^2}{(1-tz)(1-2z)(1-t)}$$
$$\times\left(\frac{(z-1)(\mathcal{G}(z,1)-\mathcal{G}(z,\frac{1}{1-z})}{z} - \frac{t(z-1)(\mathcal{G}(z,t)-\mathcal{G}(z,\frac{1}{1-z})}{1-t+tz}\right).$$

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REMARK:

All the known 2 by 4 classes have algebraic generating functions, while the three which are not known are conjectured to be non-D-algebraic.

Future questions

1. When are permutation classes amenable to this structural analysis?
2. How do we deal with functional equations where $G(z, t)$ depends on $G(z, f(z))$? Or $G(z, G(z, 1))$?
3. Are there ways to improve the guessing algorithm?

Thank you!