The Pinnacle Set of a Permutation

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Robert Davis
joint with Sarah Nelson, T. Kyle Petersen, and Bridget E. Tenner

Michigan State University

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Peaks

Definition

Let $w = w(1)w(2)\cdots w(n) \in S_n$. The peak set of w is

$$\operatorname{Pk}(w) = \{i \in \{2, \dots, n-1\} \mid w(i-1) < w(i) > w(i+1)\}.$$

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If $w = 3417625$, then $\mathrm{Pk}(w) = \{2, 4\}$.

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Theorem (Billey, Burdzy, Sagan, 2012)

If S is an admissible peak set, then

$$\#\{w \in S_n \mid \mathrm{Pk}(w) = S\} = \mathrm{pk}_S(n) 2^{n-\#S-1}$$

for some polynomial pk_S, called the peak polynomial of S, such that $\mathrm{pk}_S(k) \in \mathbb{Z}$ for any $k \in \mathbb{Z}$.

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The peak polynomial for $S = \{2, 4\}$ is

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Peak Polynomial Theorem

Theorem (Diaz-Lopez, Harris, Insko, Omar, 2016)

There exist nonnegative integers c_k^S, $0 \leq k < \max S$, such that

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The peak polynomial for $S = \{2, 4\}$ is

$$\mathrm{pk}_S(n) = 0\binom{n-4}{0} + 4\binom{n-4}{1} + 4\binom{n-4}{2} + 1\binom{n-4}{3}.$$

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2. For all n, $p_{\emptyset}(n) = 2^{n-1}$.
3. If $\max S \le t \le n$, then $p_S(n) = 2^{n-t} p_S(t)$.

Theorem (D., Nelson, Petersen, Tenner)

Let $S \subset \mathbb{Z}_{>0}$ with $\max S = m$. Then S is an admissible pinnacle set if and only if both

1. $S \setminus \{m\}$ is an admissible pinnacle set, and
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Moreover, there are $\binom{m-2}{\lfloor m/2 \rfloor}$ admissible pinnacle sets with maximum m, and

$$\binom{n-1}{\lfloor (n-1)/2 \rfloor}$$

admissible pinnacle sets $S \subseteq [n]$.

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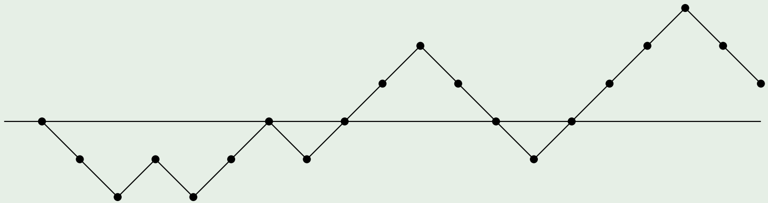
Proof sketch

Construct the set of diagonal lattice paths (with up-steps $(1,1)$ and down-steps $(1,-1)$) starting at $(0,0)$ and ending at $(n-1, n-1 \bmod 2)$.

Example ($n = 20$)



Proof sketch

Mark the up-steps strictly below the x-axis and the down-steps weakly above the x-axis.

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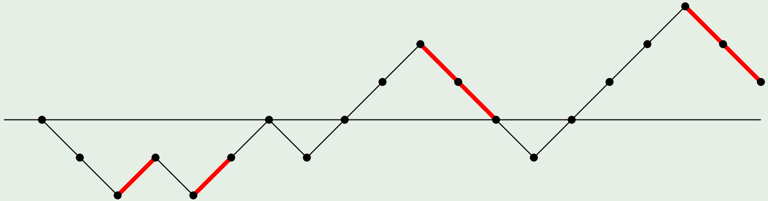

Proof sketch

If step i is marked, label it $i+1$.

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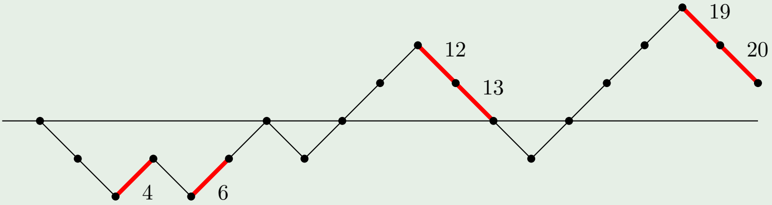

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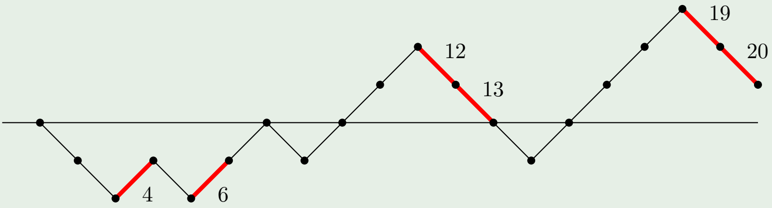

Construct the permutation $u_1 v_1 u_2 v_2 \cdots \in S_n$ where the subsequences $u_1 u_2 \ldots$ and $v_1 v_2 \ldots$ are increasing and the v_i are the labels on the marked paths.

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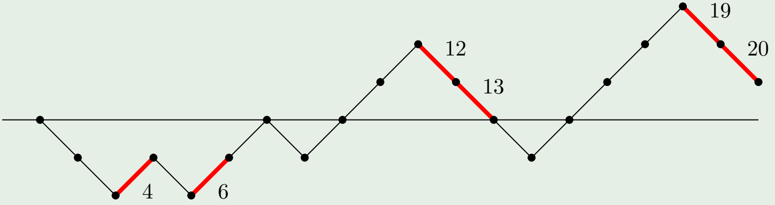

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In our example, we get the permutation

$$1, 4, 2, 6, 3, 12, 5, 13, 7, 19, 8, 20, 9, 10, 11, 14, 15, 16, 17, 18.$$

Diagonal Lattice Paths

Corollary

If $m = 2\#S + 1$, then the number of admissible pinnacle sets with maximum m and size $\#S$ is the Catalan number

$$\frac{1}{\#S+1}\binom{2\#S}{\#S}.$$

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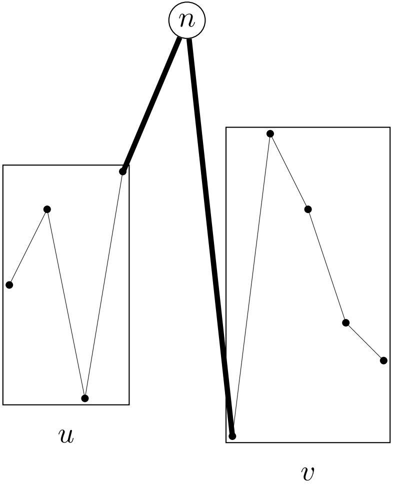
Note: a simpler proof does exist, but does not make the connection with lattice paths obvious.

Computing $p_S(n)$: a quadratic recurrence



Computing $p_S(n)$

Proposition (D., Nelson, Petersen, Tenner)

Let l, m be integers satisfying $3 \leq l < m$. For any $n \geq l$,

$$p_{\{l\}}(n) = 2^{n-2}(2^{l-2} - 1)$$

and for any $n \geq m$,

$$p_{\{l,m\}}(n) = 2^{n+m-l-5}\left(3^{l-1} - 2^l + 1\right) - 2^{n-3}(2^{l-2} - 1).$$

Computing $p_S(n)$: a linear recurrence

Proposition (D., Nelson, Petersen, Tenner)

Suppose S is an admissible pinnacle set and $\max S = m$. For any $n \geq m$, $p_S(n)$ is given by

$$2^{n-m}\left((m-2\#S)p_{S\setminus\{m\}}(m-1)+2\sum_{\substack{T=(S\setminus\{m\})\cup\{j\}\\ j\in[m]\setminus S}} p_T(m-1)\right).$$

Bounds on $p_S(n)$ for fixed size of S

Theorem (D., Nelson, Petersen, Tenner)

Let S be an admissible pinnacle set of size d, and suppose $n > 2d$. Then the following bounds are sharp:

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where $S(\cdot, \cdot)$ denotes the Stirling number of the second kind.

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The lower bound is achieved by choosing $S = \{3, 5, \ldots, 2d+1\}$ and applying the linear recurrence.

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The upper bound is achieved by choosing $S = \{n-d+1, n-d+2, \ldots, n\}$.

Maximizing $p_S(n)$ over all $S \subseteq [n]$

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For fixed n, let $d(n) = k < n/2$ be the value maximizing the expression $k!(k+1)!2^{n-2k-1}S(n-k, k+1)$.

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$d(n)$	1	1	1	2	2	2	3	3	3

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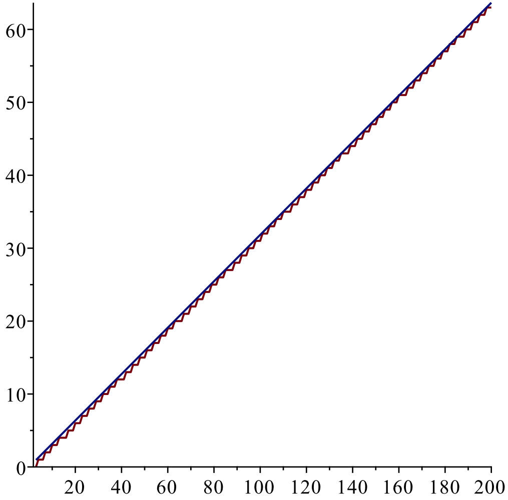
n	4	5	6	7	8	9	10	11	12	13	14	15	16
$d(n)$	1	1	1	2	2	2	3	3	3	4	4	4	4

Plot: $d(n)$ for $n \leq 200$ vs. the line $k = n/\pi$



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Question

For general n and S, is there a closed-form, non-recursive formula for $p_S(n)$?