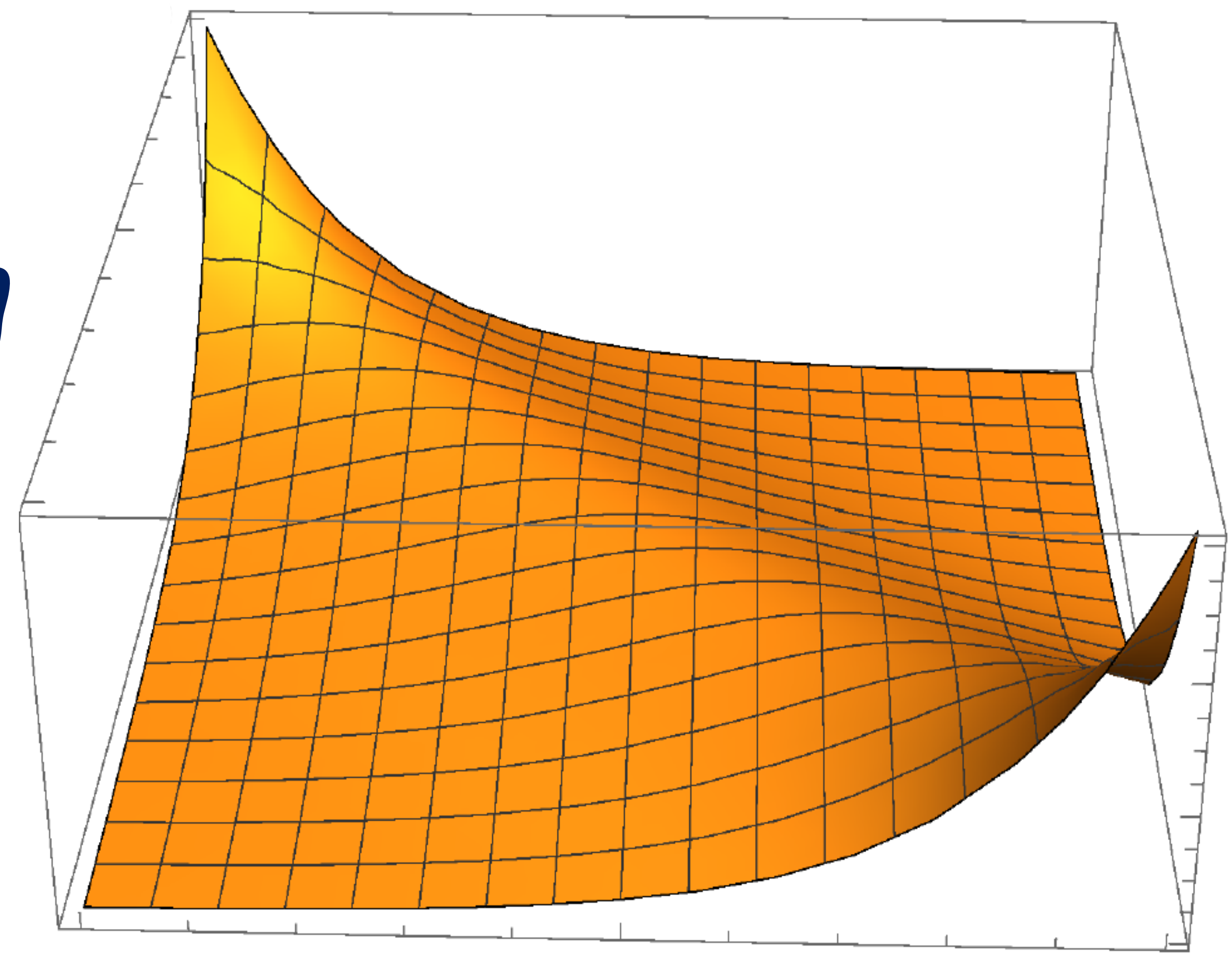
Permutation Patterns, Reykjavik 6/17

Permutons and Pattern Densities



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with

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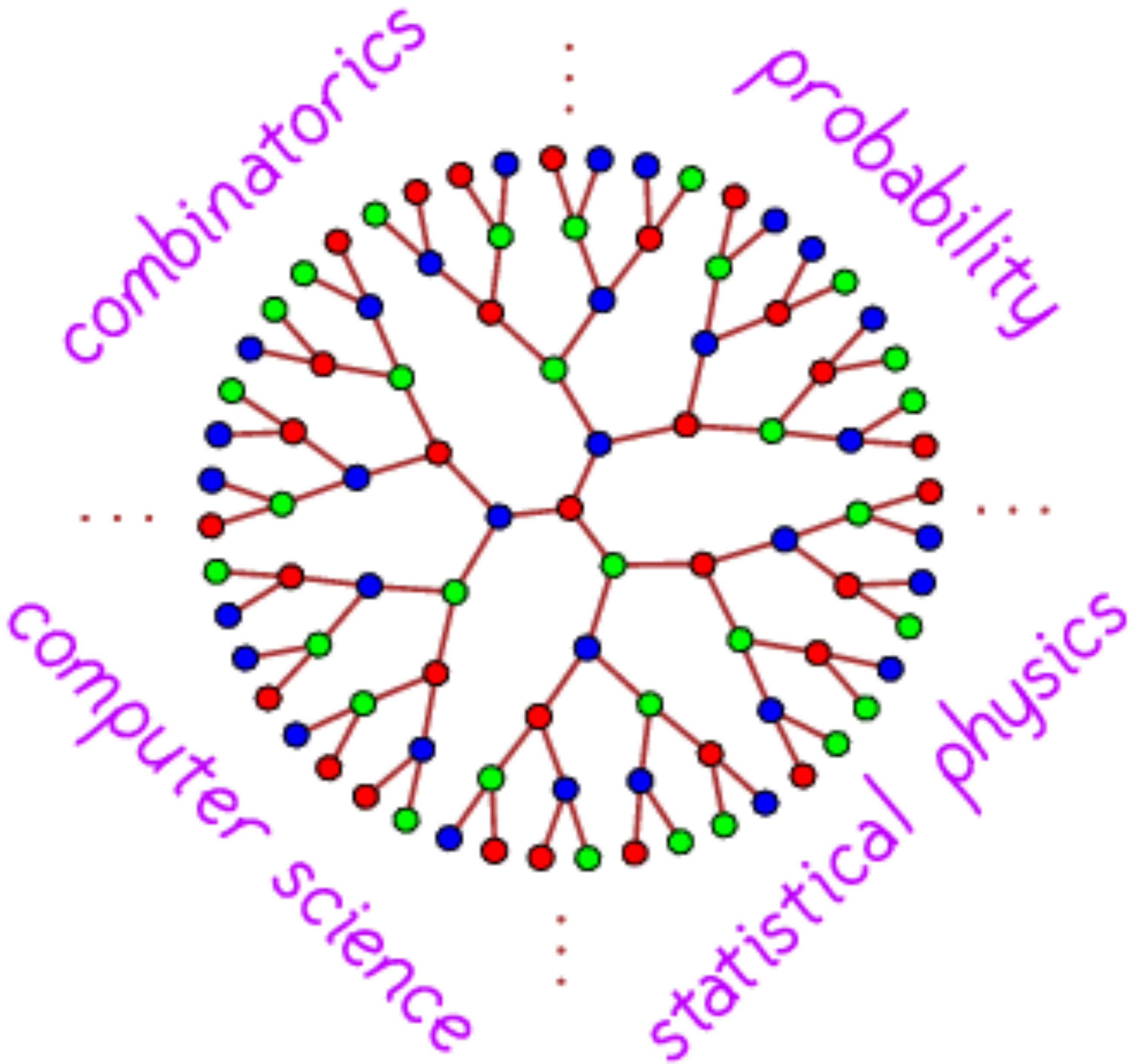
work begun at ICERM, spring 2015



Common observation:

Large random objects tend to look alike.

Common problem:

What ***do*** they look like?

Common approach:

Count them and take limits.

Today's large random objects:

permutations of $\{1,\dots,n\}$ for large n
(or those with some given property).

What sort of "gross" property do we care about for large permutations?

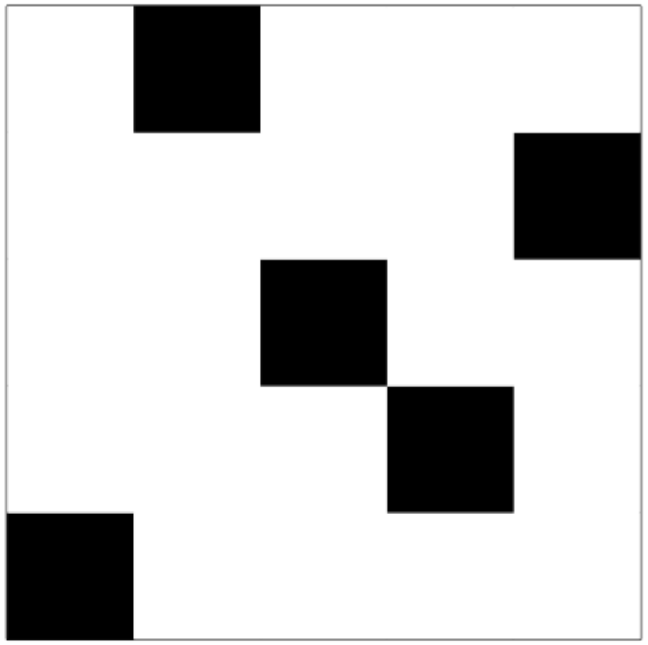
Perhaps pattern densities?

Pattern density $\rho_\pi(\sigma) :=$
occurrences in σ of the pattern π,
divided by $\binom{n}{k}$

A *permuton* is a probability measure on $[0,1]^2$ with uniform marginals (AKA doubly-stochastic measure, or two-dimensional ***copula***).

permutation
15324

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

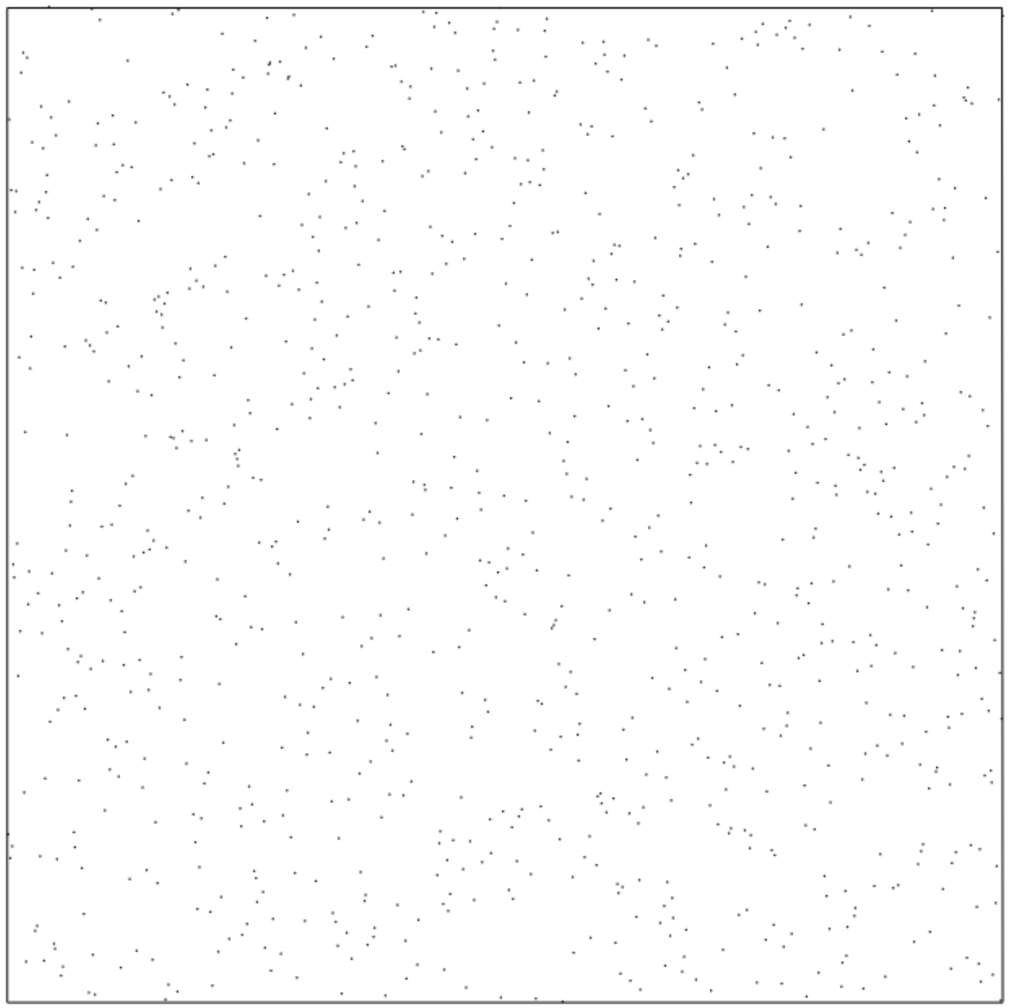


density 5

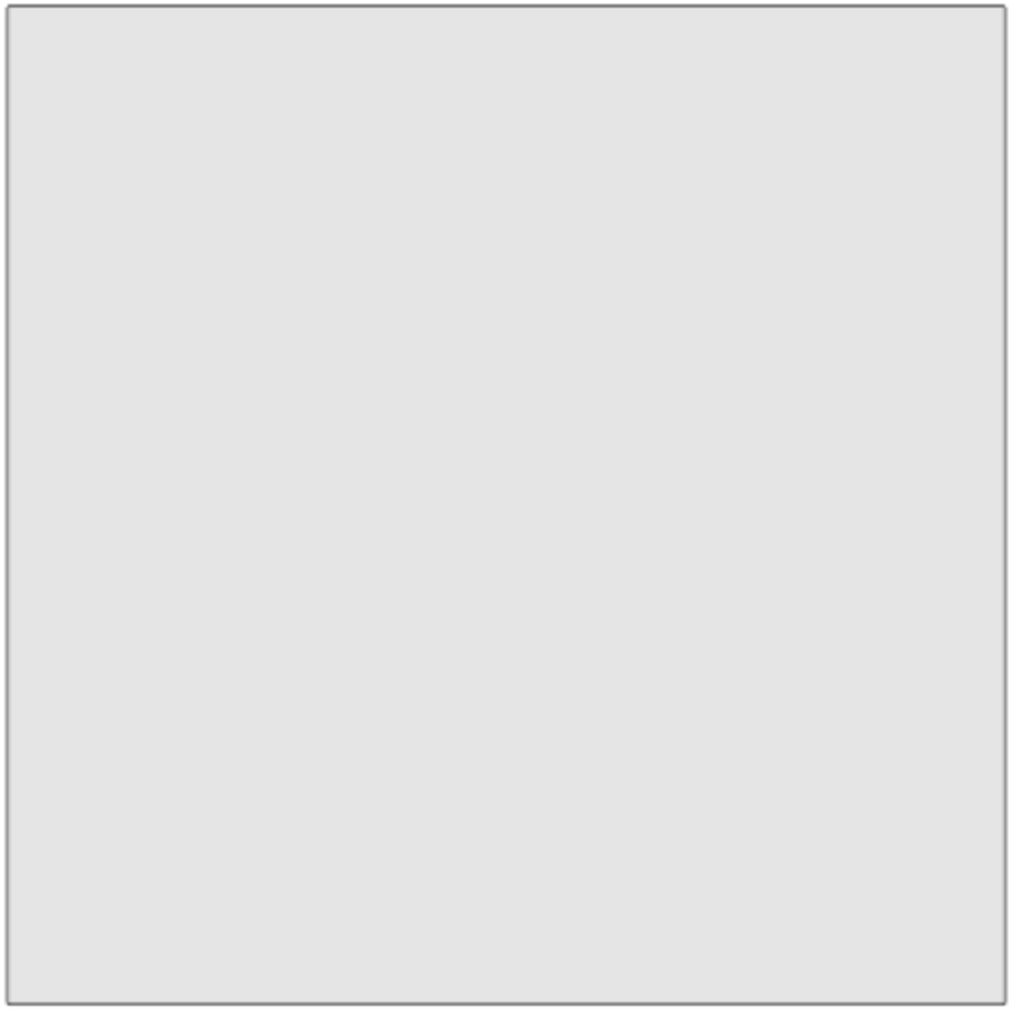
permuton
$\gamma(15324)$

Every permutation σ provides a corresponding permuton $\gamma(\sigma)$.

"urban" permuton $\gamma(\sigma)$ for a random σ in S_{1000}



uniform permuton

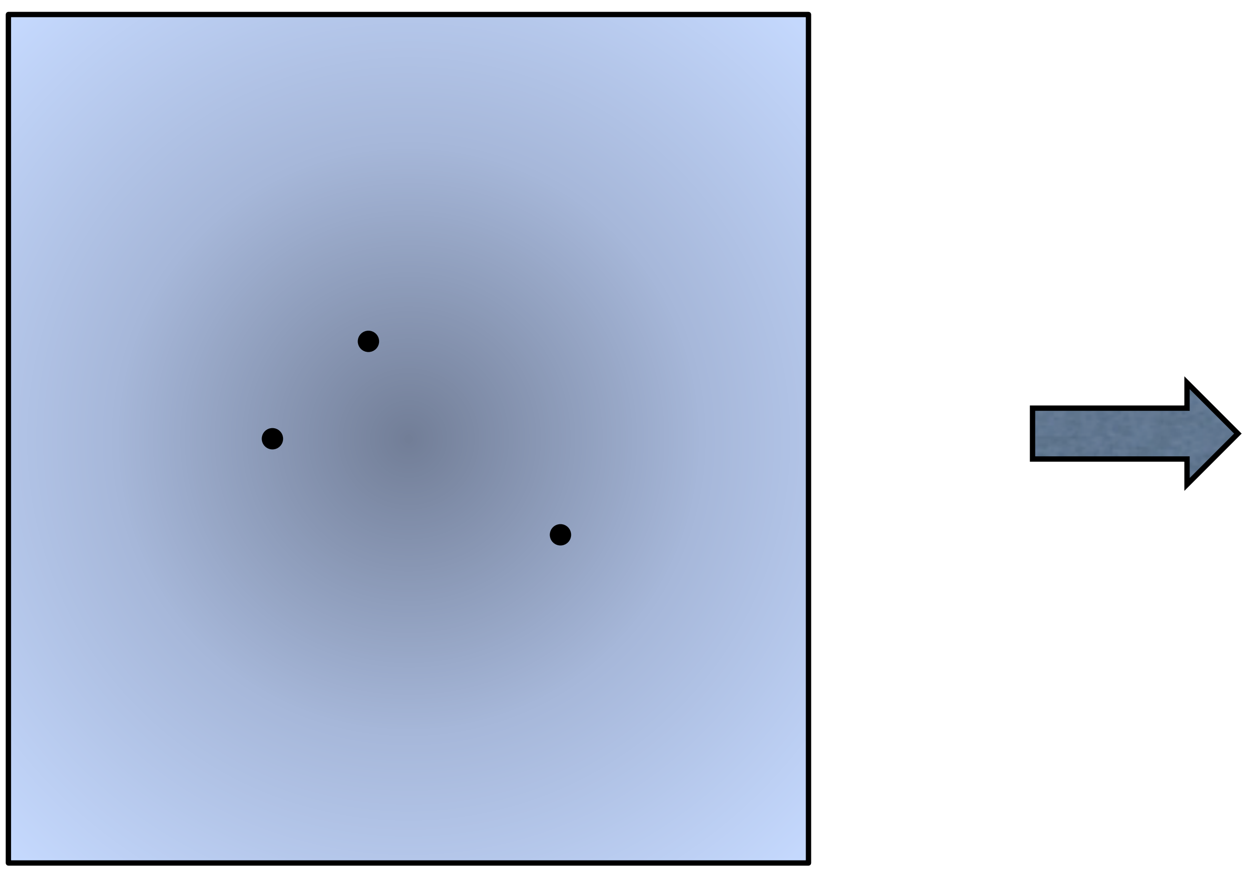


A sequence of permutations *converges* if their permutons converge *in distribution*, i.e., their CDF's converge pointwise.

The CDF of γ is $G(x,y) := \gamma([0,x]\times[0,y])$.

To each permuton γ is associated a probability measure γ_n on S_n:

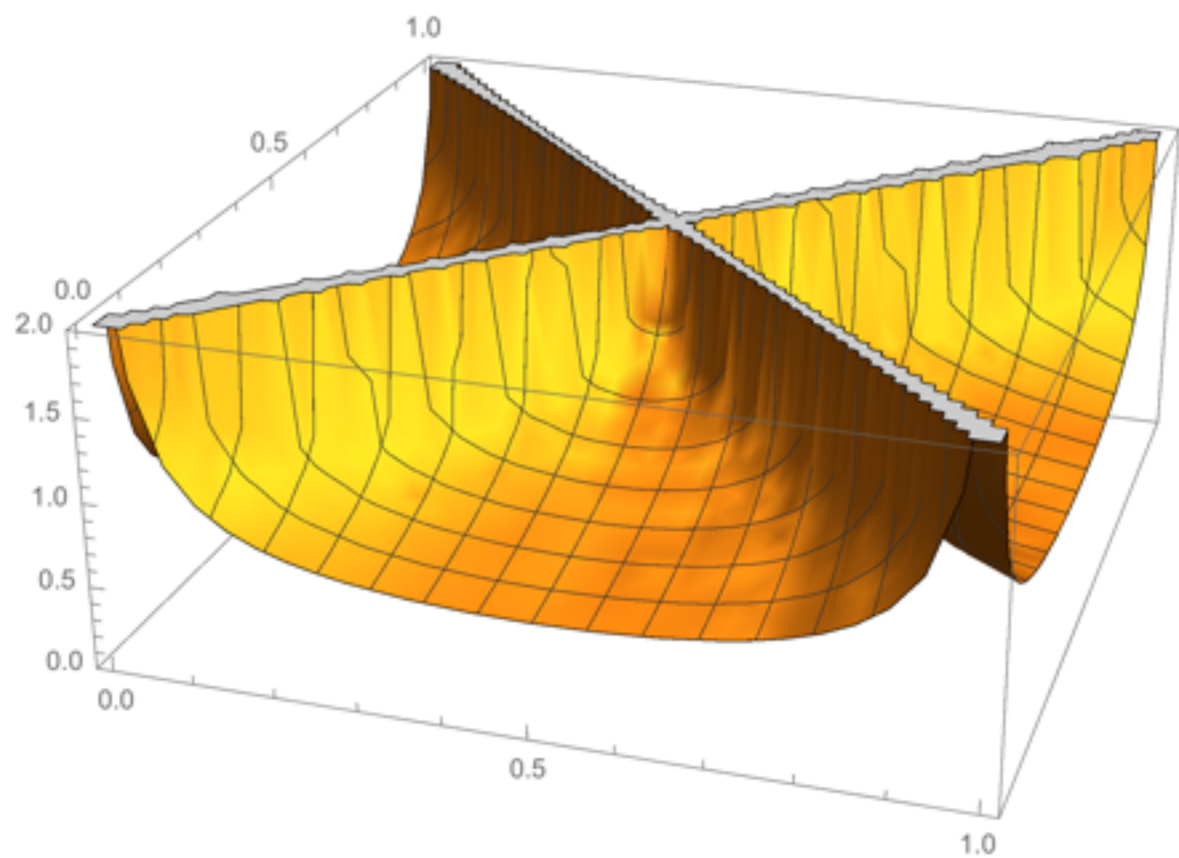
1. Pick n i.i.d. points from γ
2. Sort them by x-coordinate
3. Record the permutation given by the y-coordinates.



231

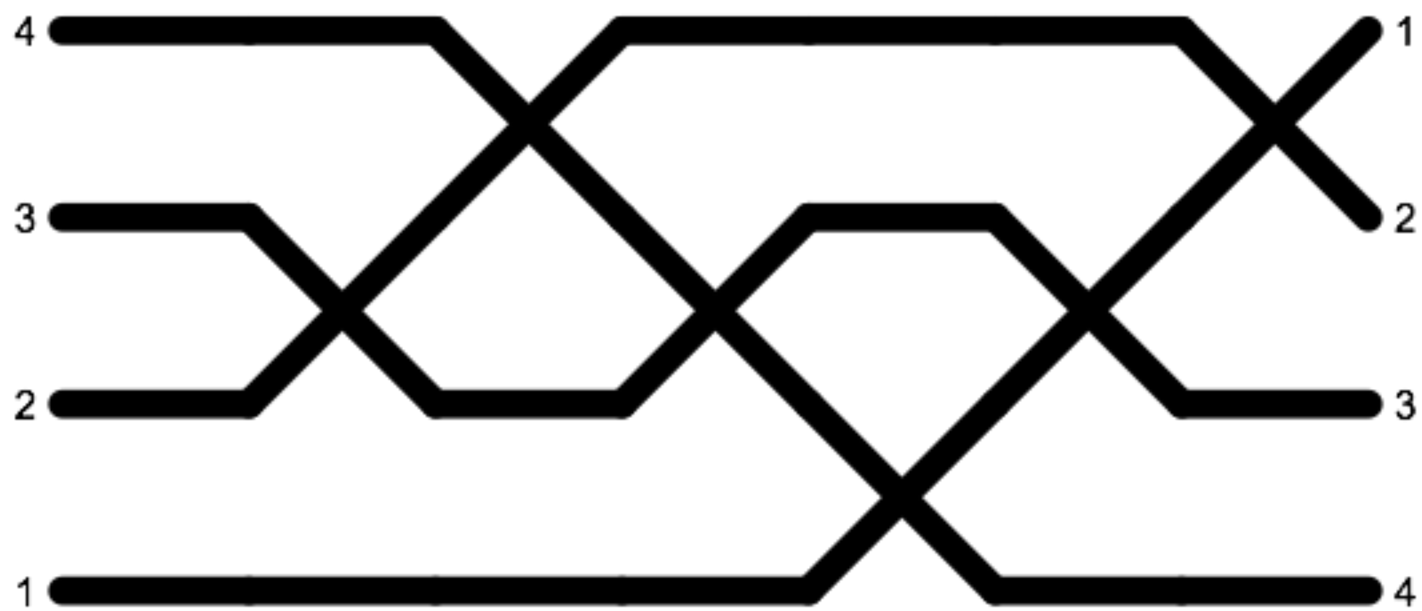
Permutons for some naturally arising measures

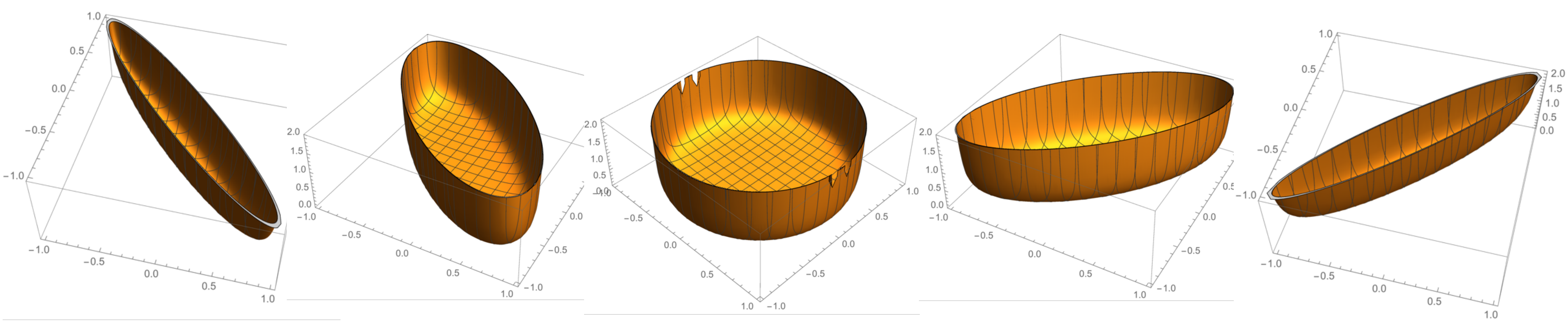
Take n-1 steps of a random walk on the real line
With symmetric, continuous step distribution, and
let π_n be the induced permutation on values.



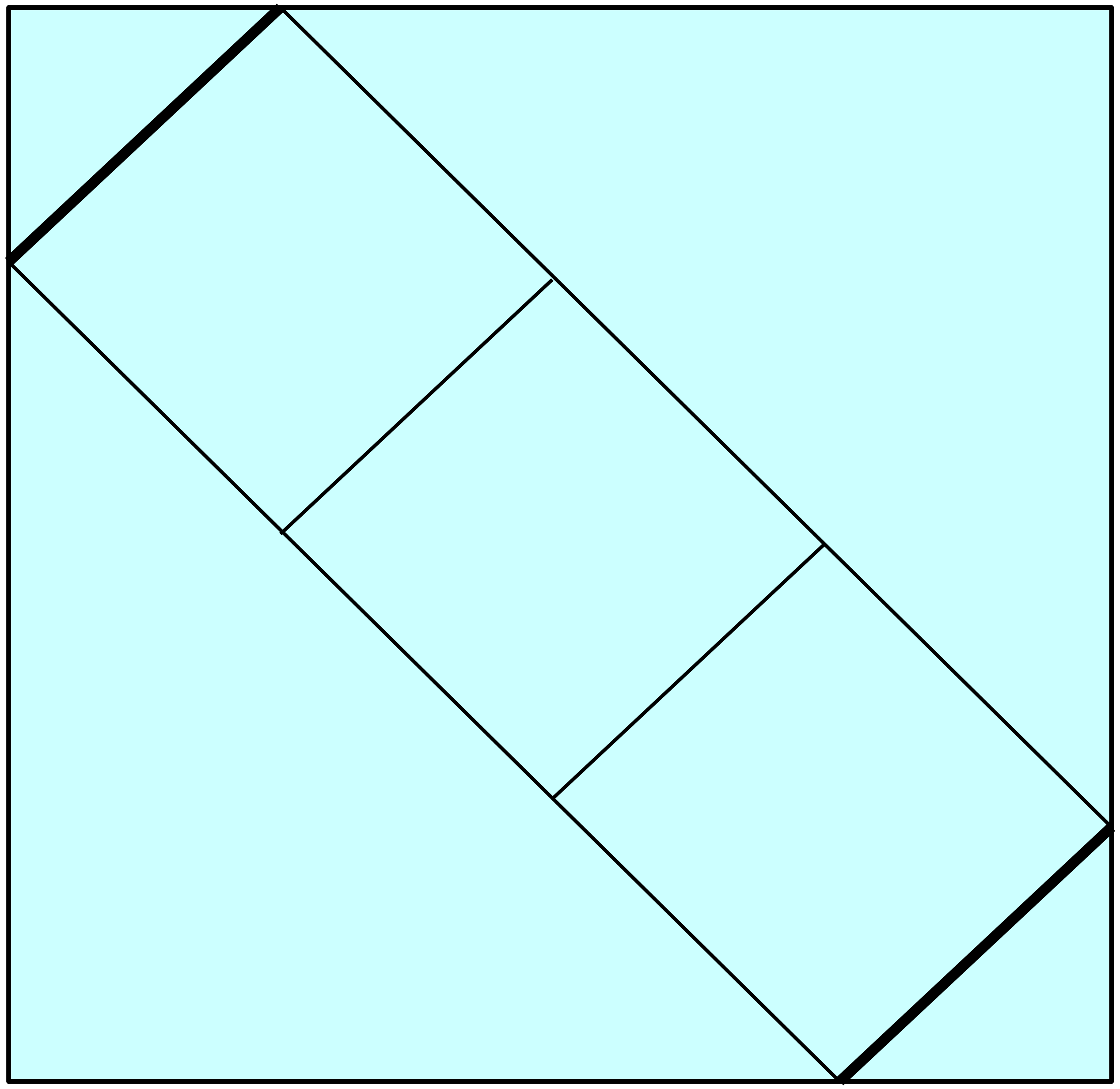
$$g(x,y) = \frac{1}{\pi^2}\int_0^x \frac{du}{\sqrt{u(x-u)(y-u)(1-x-y+u)}}.$$

Permutons that conjecturally describe permutations encountered at stages of a random sorting network:





A singular permuton



(in this case: a 1324-avoiding graphical grid class)

The density of a pattern π of length k in a permuton γ is just $\gamma_k(\pi)$.

For example, the 21-density, AKA the *inversion density* of γ, is

$$2\int_{u<x}\int_{v>y} g(u,v)g(x,y)\,du\,dx\,dv\,dy$$

provided γ is lucky enough to have a density g.

Although $\rho(\sigma)$ is not exactly equal to $\rho(\gamma(\sigma))$,

Thm [Hoppen, Kohayakawa, Moreira, Rath & Sampaio '13]:

(1) A permuton is determined by its pattern densities;

(2) Permutons are the completion of permutations in the (metric) pattern-density topology.

We wish to study subsets of S_n of size

$n!e^{cn}$, that is, $e^{n \log n - n + cn}$,

where c is some non-positive constant.

Example: Permutations with one or more pattern densities fixed.

But: If one of those densities is 0, we know from the Marcus/Tardos '04 proof of the Stanley-Wilf conjecture that the class is "only" exponential in size.

The entropy of γ_n is

$$\mathrm{ent}(\gamma_n) = \sum_{\pi \in S_n} -\gamma_n(\pi) \log \gamma_n(\pi)$$

Example: the entropy of the uniform distribution on S_n is $\log n!$.

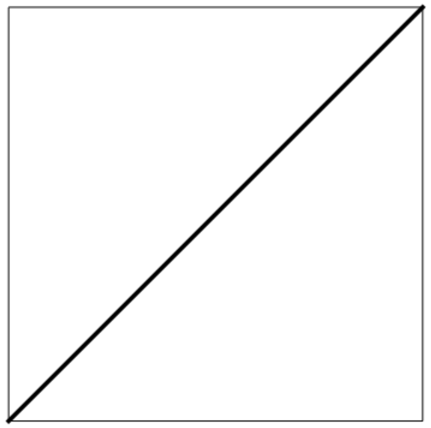
Definition: the permuton entropy is

$$H(\gamma) := \lim_{n \to \infty} \frac{1}{n}\left(\mathrm{ent}(\gamma_n) - \log n!\right)$$

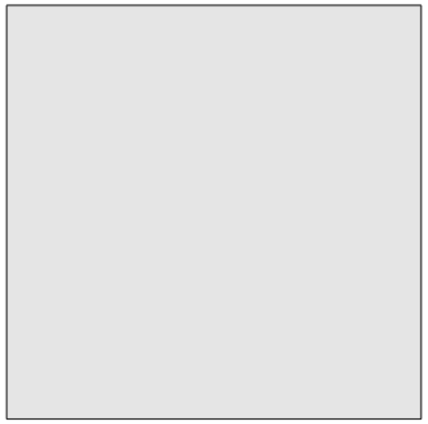
Thm: $H(\gamma) = \int\int -g(x,y) \log g(x,y)\, dx\, dy$

with $H(\gamma) = -\infty$ if $g \log g$ is not integrable or γ has no density.

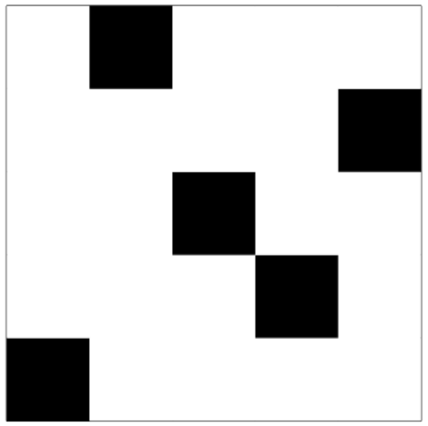
Sample entropies



$H = -\infty$



$H = 0$



$H = -\log 5$

Permuton entropy is never positive,
and $= 0$ only for the uniform measure.

Large deviations principle:

(various versions and proofs due to Trashorras '08, Mukherjee '15, and KKRW '15.)

Thm: Let Λ be a "nice" set of permutons, with $\Lambda_n = \{\pi \in S_n : \gamma(\pi) \in \Lambda\}$. Then

$$\lim_{n \to \infty} \frac{1}{n} \log(|\Lambda_n|/n!) = \sup_{\gamma \in \Lambda} H(\gamma).$$

Variational principle:

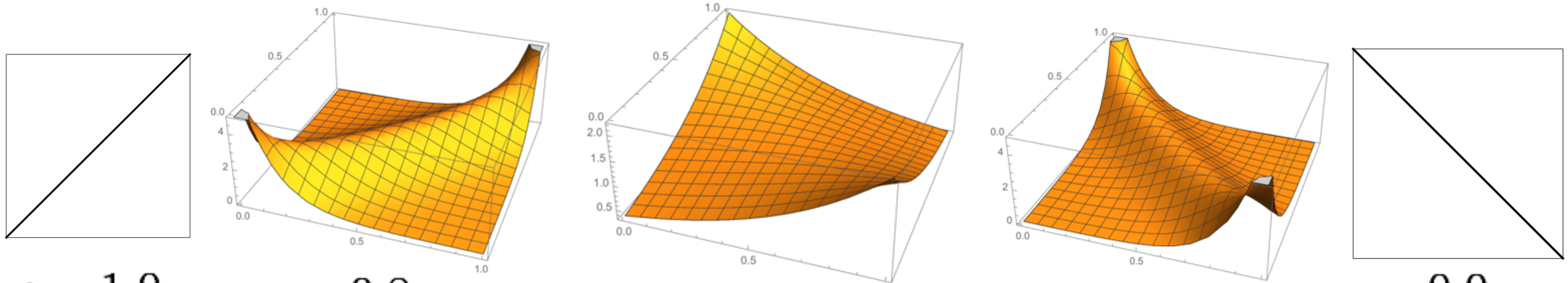
To describe and count permutations with given properties (e.g., with certain fixed pattern densities), find the permuton with those properties that *maximizes entropy*.

Example: Fix the density ρ of the pattern 12.

There are lots of permutons with density ρ of the pattern 12, but there's a unique one μ_ρ of maximum entropy.

A uniformly random permutation of $\{1,\ldots,n\}$ with density ρ of the pattern 12 will "look like" μ_ρ for large n (i.e., its permuton will be close to μ_ρ).

Permutons with fixed 12 density



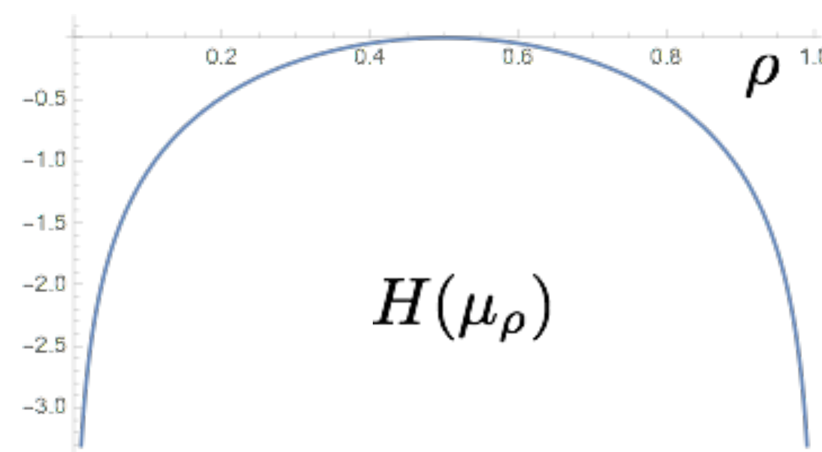
$\rho = 1.0$ 0.8 0.4 0.2 0.0

There is an explicit density for μ_ρ (see also Starr '09):

$$g(x,y) = \frac{r(e^r - 1)e^{r(x+y)}}{(e^r - e^{rx} - e^{ry} + e^{r(x+y)})^2}$$

where

$$\rho = \frac{-6\mathrm{Li}_2\left(e^r\right) + 3r(r+2) - 6r\log\left(1 - e^r\right) + \pi^2}{6r^2}.$$



Our LDP proof: mostly analysis.

One bit of combinatorics:

Baranyai's Lemma: The entries of any real matrix with integer row and column sums can be rounded to integers in such a way that the row and column sums are preserved.

$$\begin{array}{|c|c|c|}\hline 2.3 & 3.2 & 1.5 \\ \hline 2.6 & 5.2 & 3.2 \\ \hline 4.1 & 2.6 & 0.3 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|}\hline 2 & 3 & 2 \\ \hline 3 & 5 & 3 \\ \hline 4 & 3 & 0 \\ \hline \end{array}$$

Used to construct permutations that approximate a permuton with given density.

Our "inserton" approach, applied to finding the permuton for fixed 12-density:

Build random permutation inductively---for each i, insert i somewhere into the current permutation of $1,2,\ldots,i-1$.

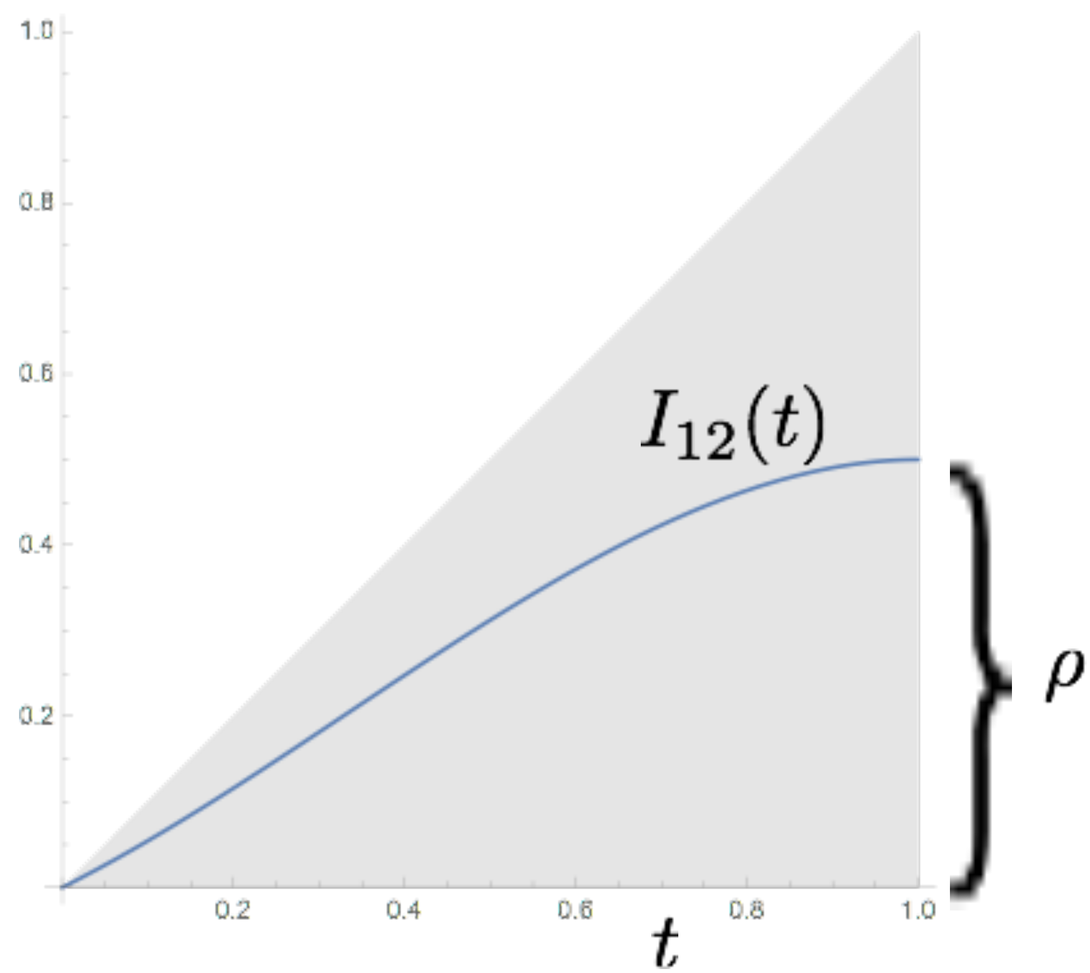
Note that if i is inserted into the j th position, we get $j-1$ more 12 patterns.

Mimic this process continuously, letting $f_t(y)dy$ be the insertion density at time t.

Lemma. The entropy of the permuton with insertion measures $f_t(y)dy$ is

$$H(\gamma) = \int\int - f_t(y) \log(t f_t(y))\, dy\, dt.$$

Let $I_{12}(t)$ be the number of 12 patterns after time t.



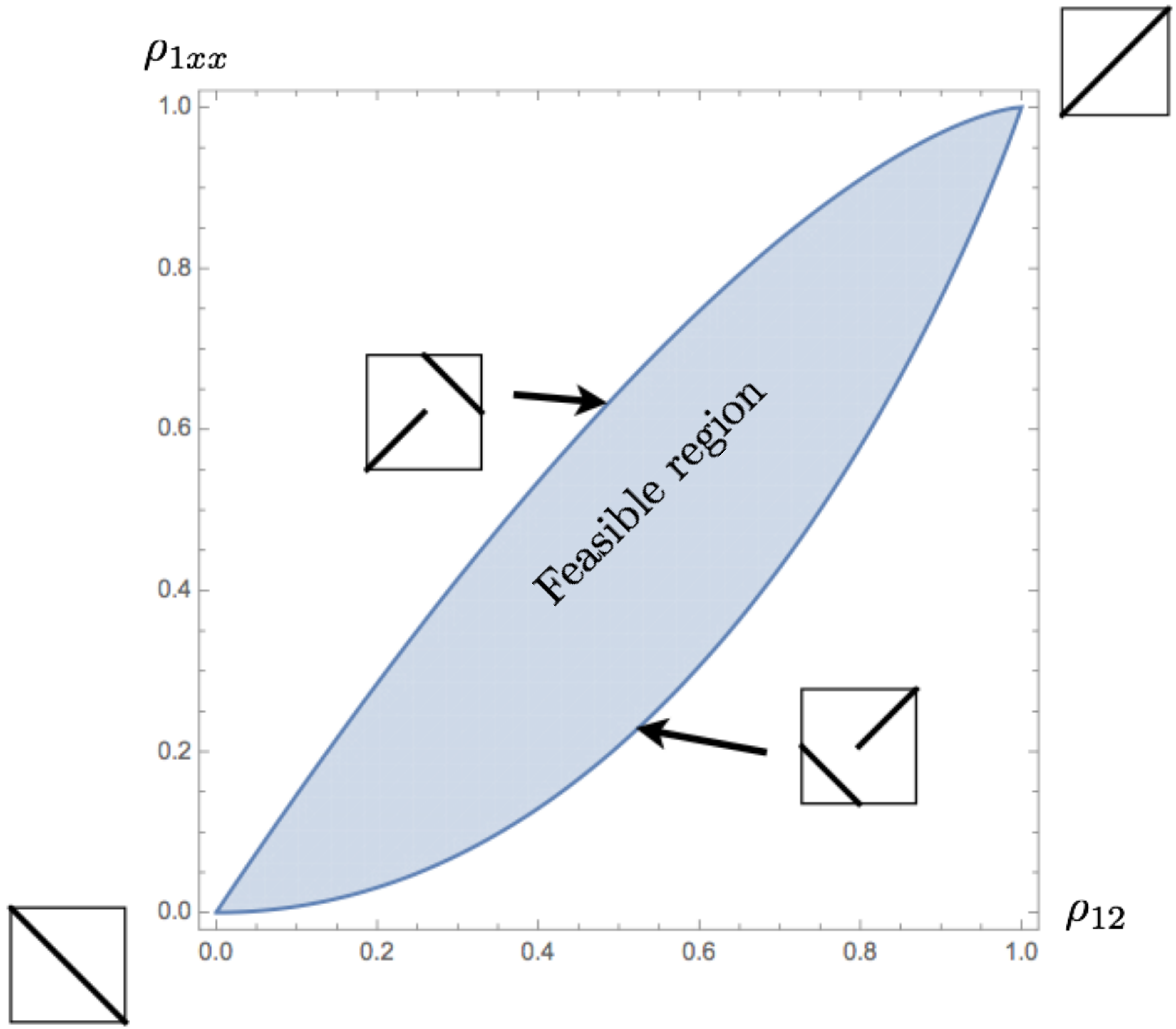
Then $I'_{12}(t)$ is the mean insertion location at time t.

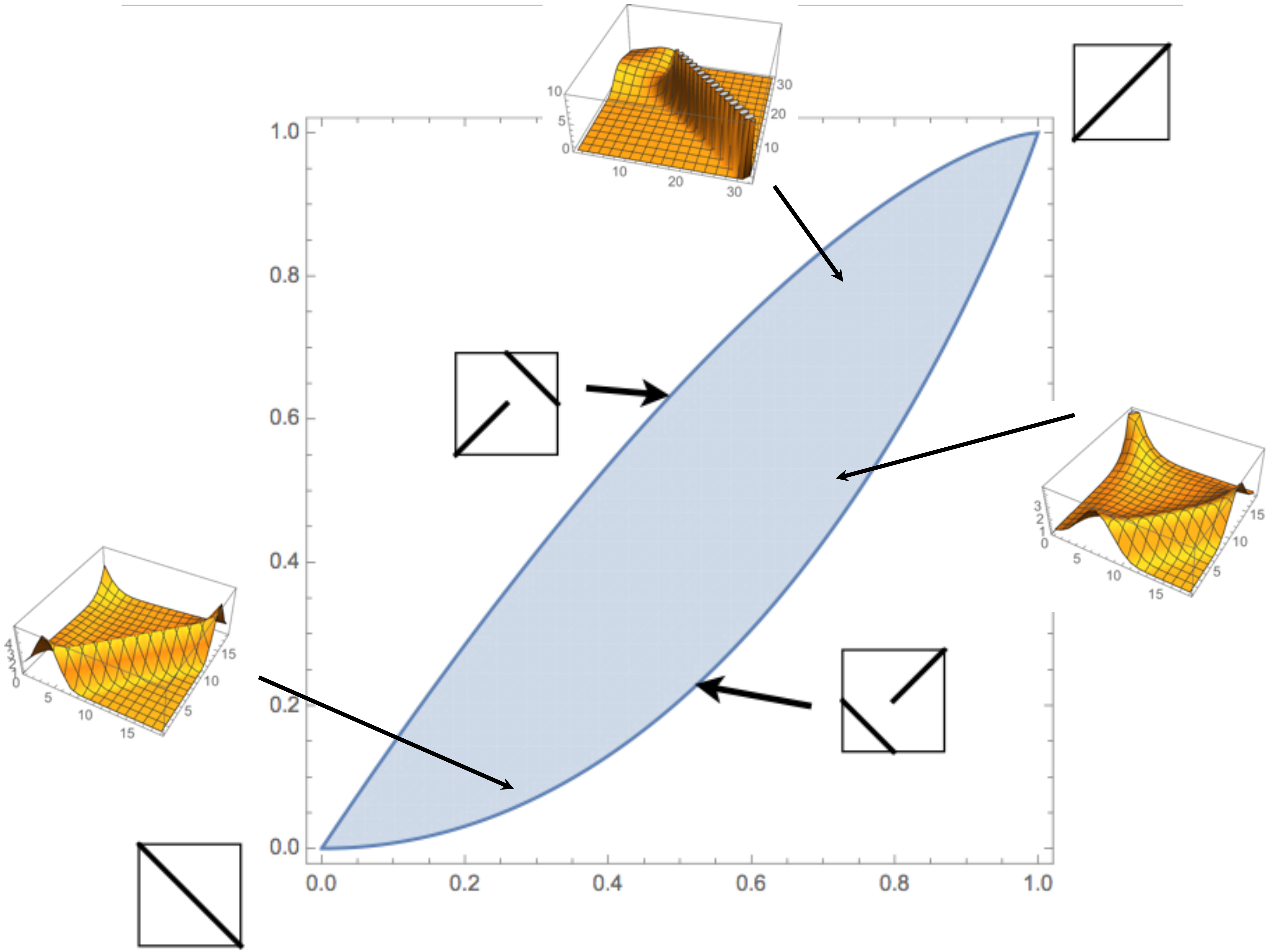
To maximize $H(\gamma)$ for fixed $\rho = I_{12}(1)$,

1. Take f_t to be a truncated exponential (maximizing its entropy for fixed mean);

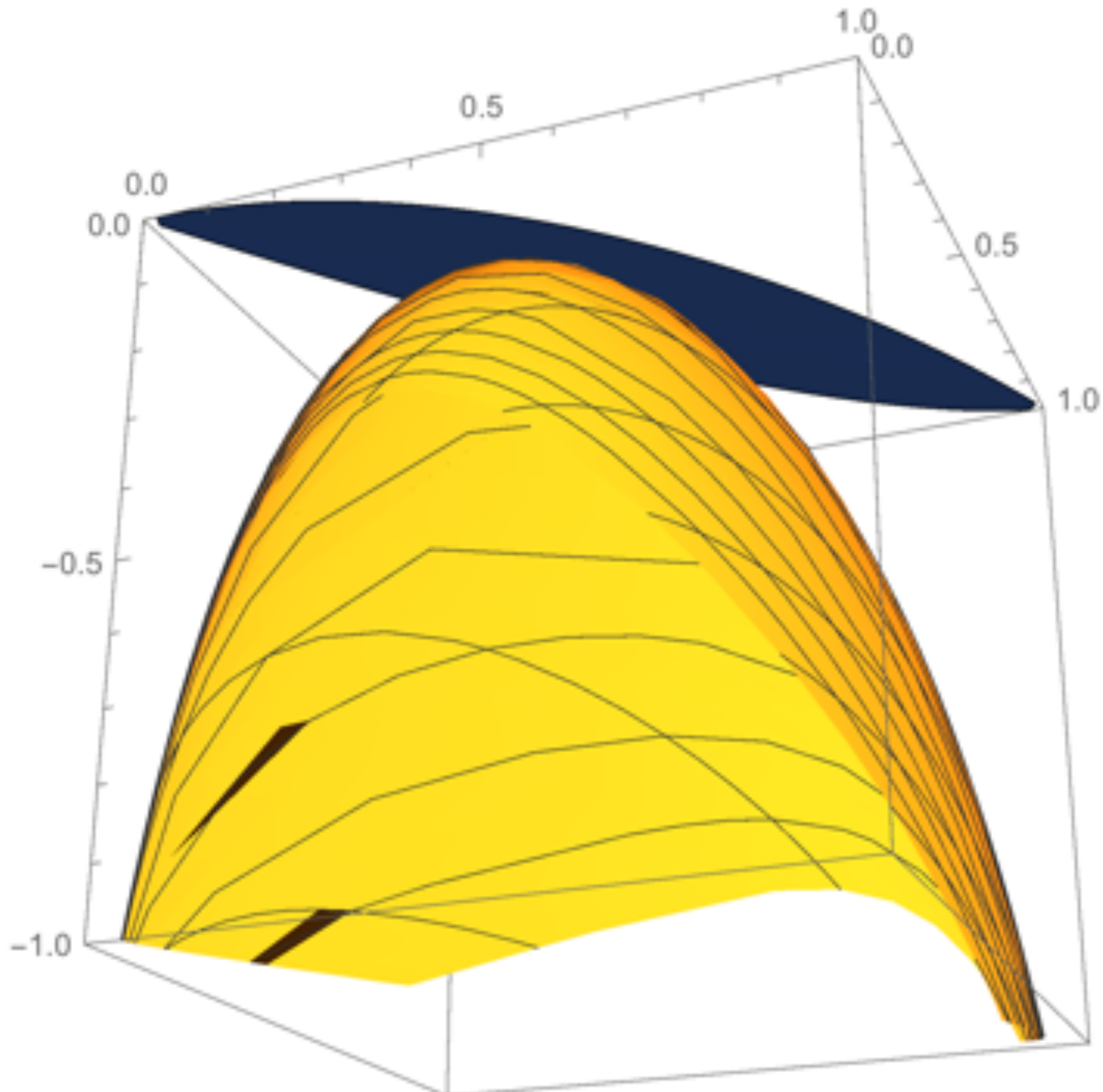
2. Take $I'_{12}(t) = \text{const}$ (so all f_t have same rate).

Fix densities of 12 and 1¤¤ (= 123 + 132):





1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0



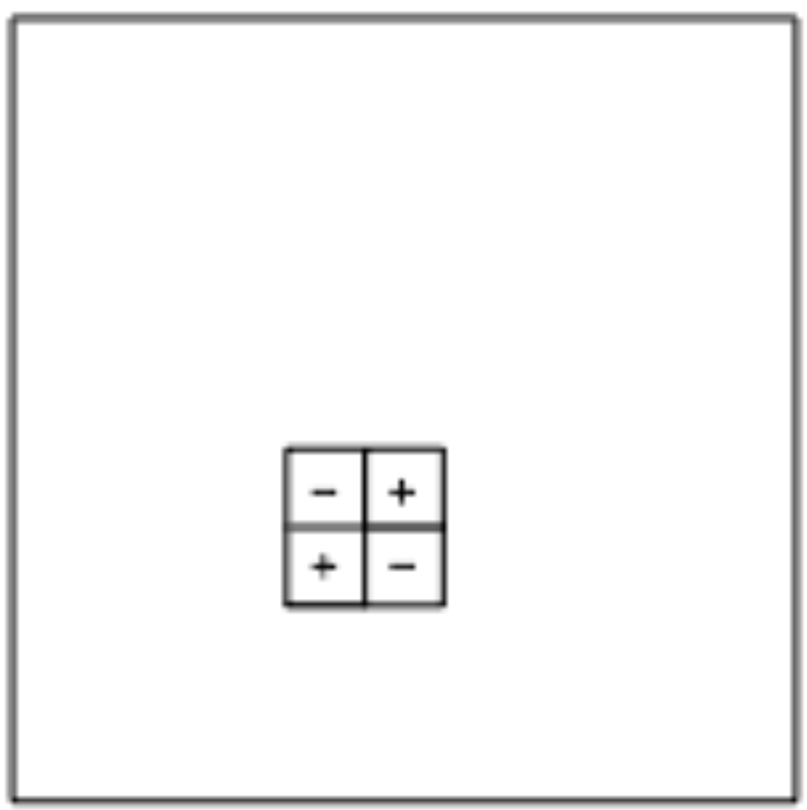
Concavity of the entropy function helps make this space solvable.

In dealing with other short patterns:

Thm: The maximizing permutons for any patterns of length 2 or 3 satisfy a PDE of the form

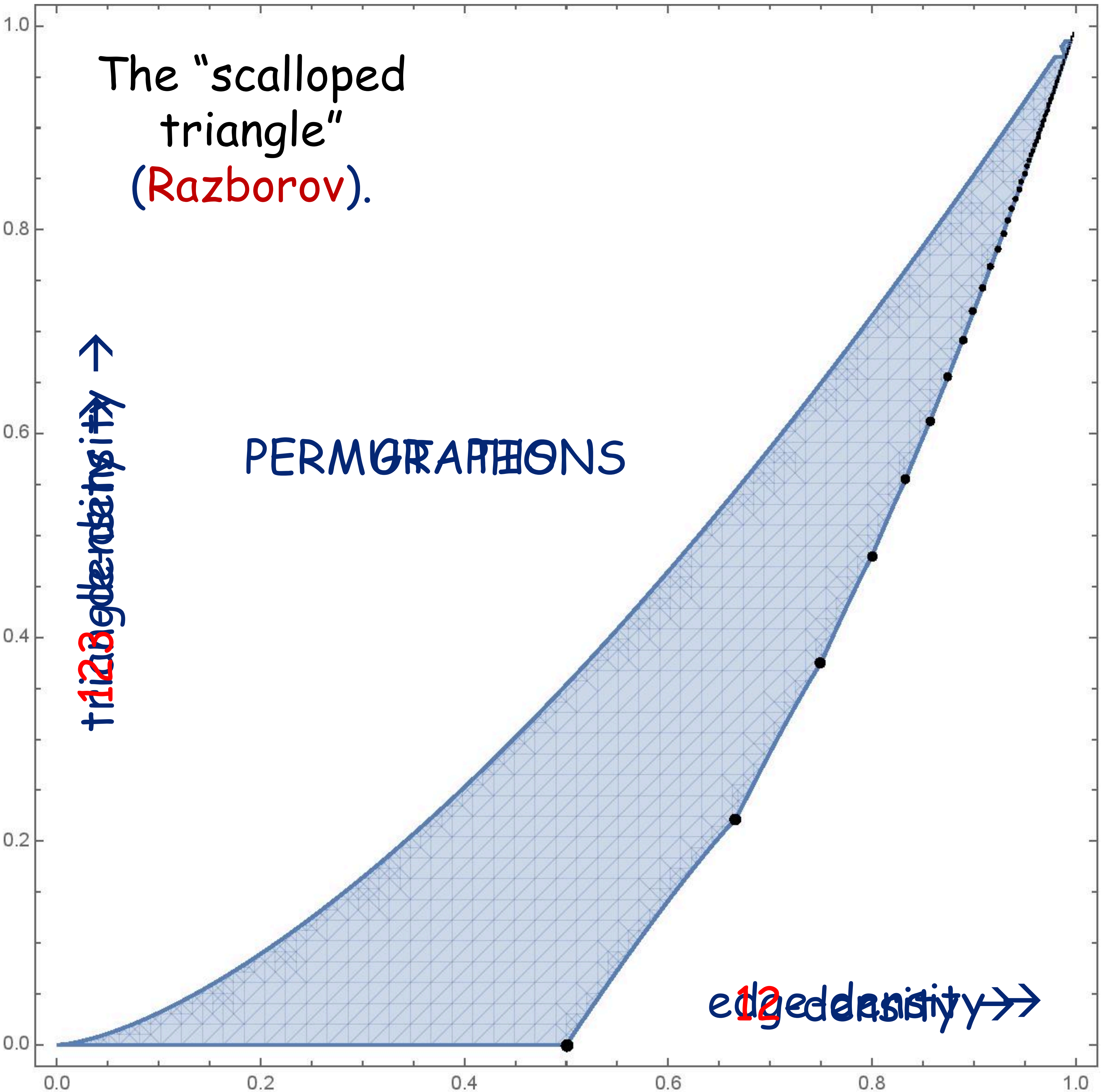
$$(\log G_{xy})_{xy} + \beta_1(2GG_{xy} + G_xG_y) + \beta_2 = 0$$

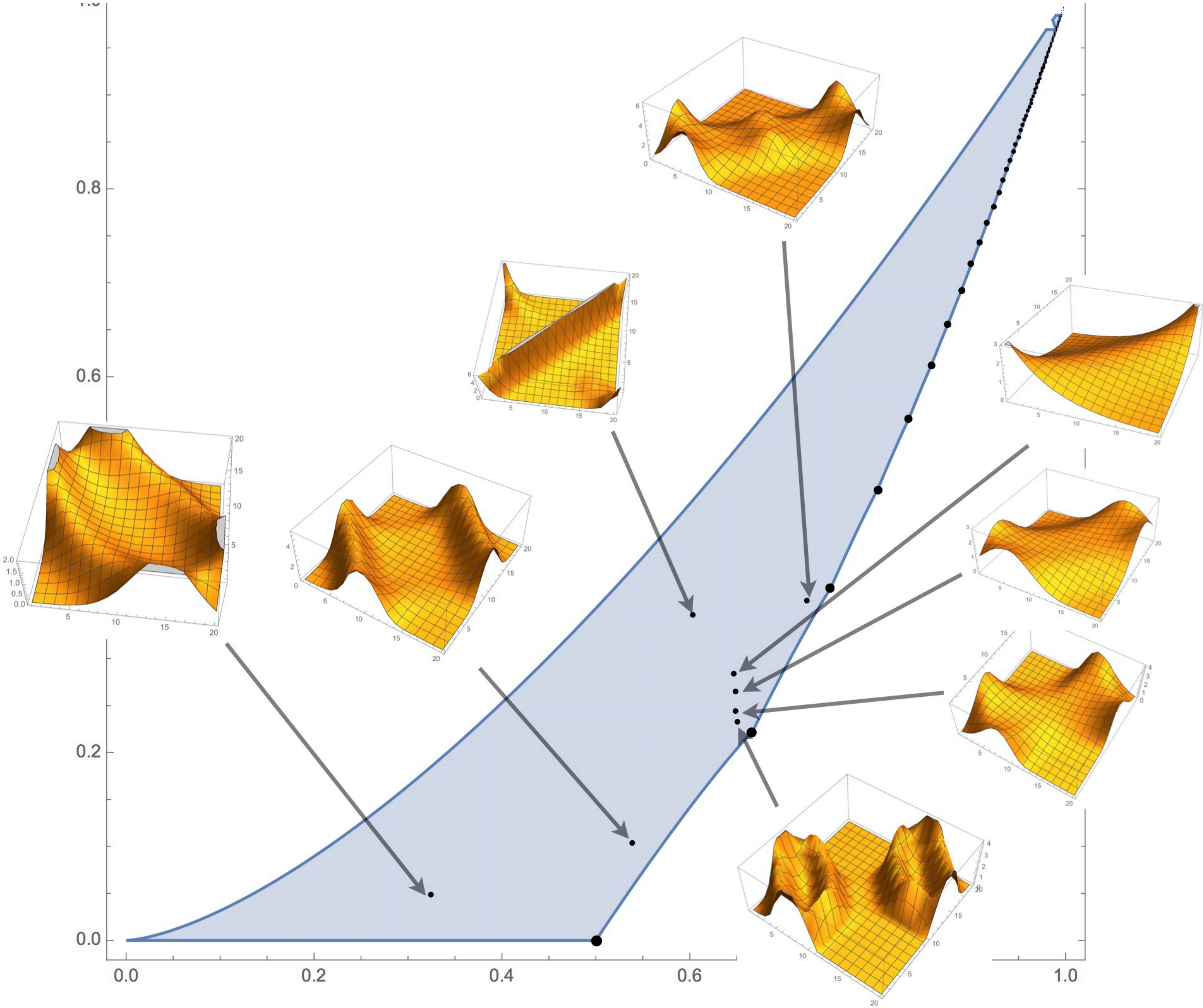
Proof idea:



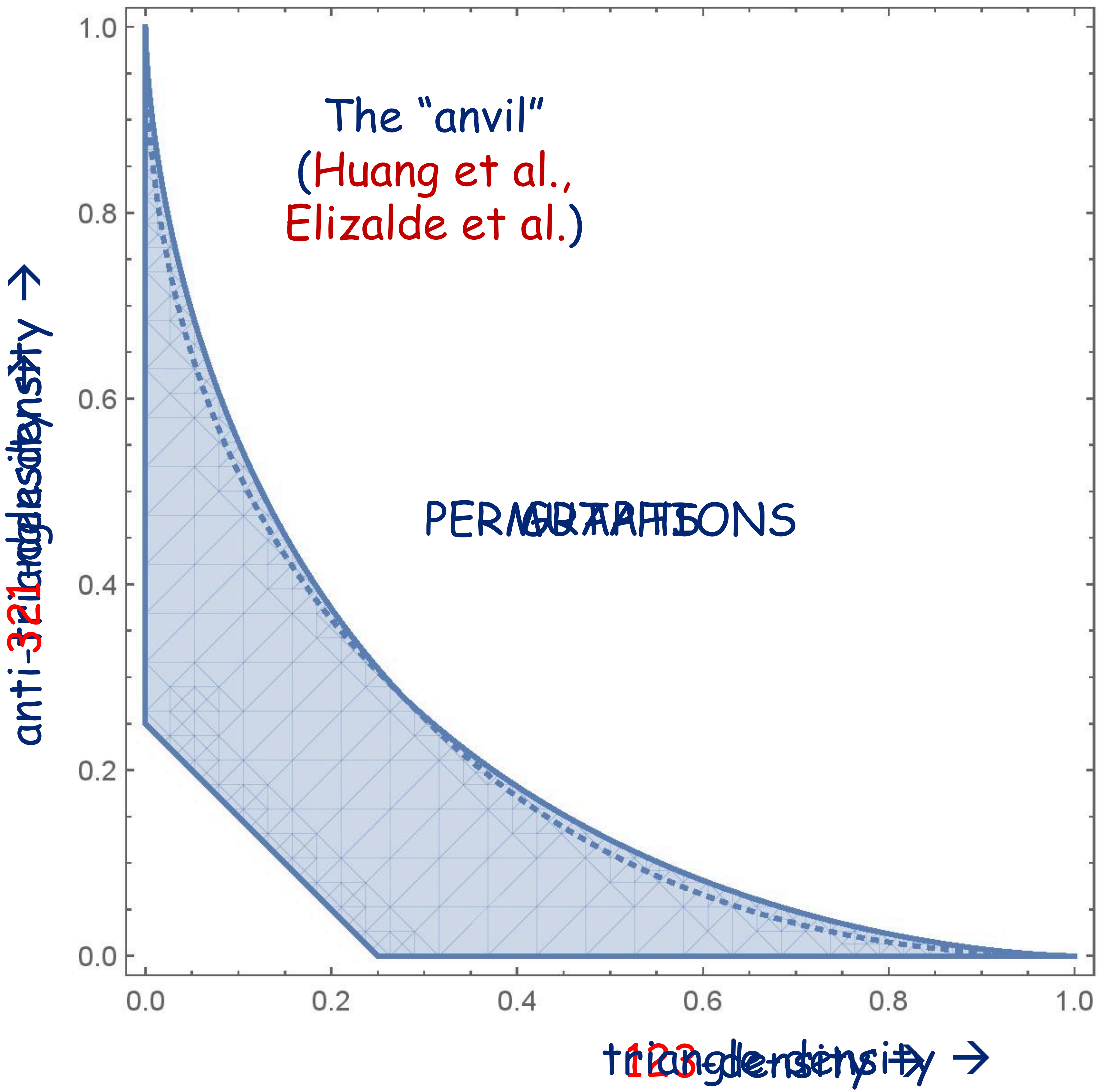
CONTRAST:
Entropy-maximizing graphons are not analytic!
(see work of Radin, Sadun +.)

Move mass around respecting marginals, so as (for example) to increase $H(\gamma) + \beta\rho_{123}(\gamma)$.

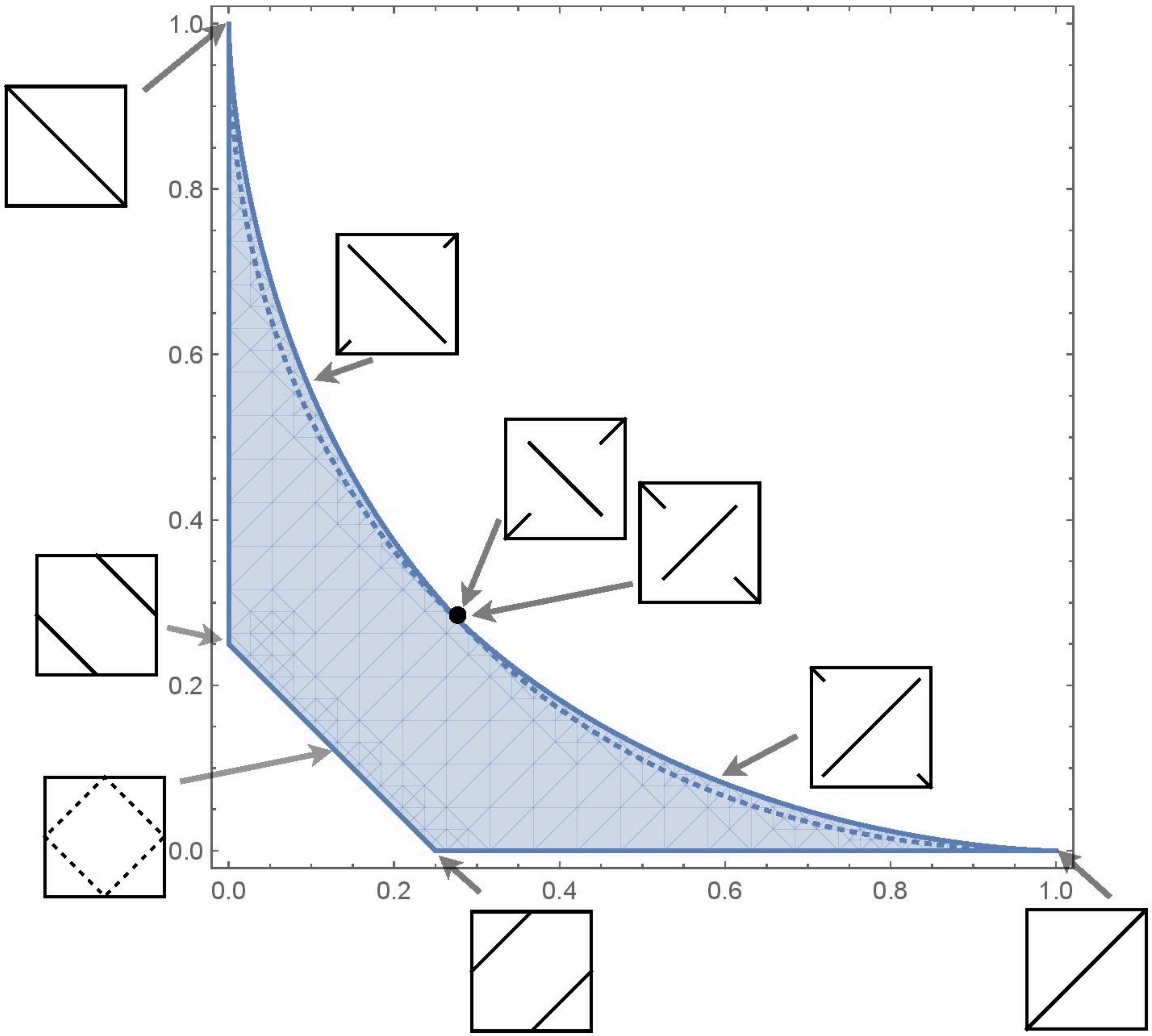

The “scalloped triangle” (Razborov).
PERMUTATIONS
GRAPHS
edge density →
12 density →
triangle density →
123 density →
0.0
0.2
0.4
0.6
0.8
1.0



0.0
0.2
0.6
0.8
0.0
0.2
0.4
0.6
1.0



The "anvil"
(Huang et al.,
Elizalde et al.)
PERMUTATIONS
GRAPHS
0.0
0.2
0.4
0.6
0.8
1.0
triangle density →
123-density →



1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Some of the (many) open questions:

Q1: Does every interior point of a feasibility region represent a large set of permutations (i.e., must it have a permuton of finite entropy?)

Q2: Does every entropy-maximizing permuton have an analytic density function?

Q3: What can be learned about avoidance classes by looking at limits of entropy-maximizing permutons as you approach the boundary of a feasibility region?

Q4: We know that for any single fixed pattern π, the entropy of the permuton whose π-density is ρ is unimodal in ρ. But we haven't proved it's continuous!

Thank you!