Equidistributions of Mahonian statistics over pattern avoiding permutations

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Permutation Patterns, Reykjavik

Overall Point

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- Combine the theory of pattern avoidance with the theory of statistics.

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- Study the generating function

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- Example questions: Equidistribution? Recurrence relation? Unimodality/Log-concavity/real-rootedness? etc..

Our focus

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- We study equidistributions of the form

$$\sum_{\sigma \in \mathsf{Av}_n(\pi_1)} q^{\mathsf{stat}_1(\sigma)} = \sum_{\sigma \in \mathsf{Av}_n(\pi_2)} q^{\mathsf{stat}_2(\sigma)}$$

where $\pi_1, \pi_2 \in \mathcal{S}_3$ are (classical) patterns and $\mathsf{stat}_1, \mathsf{stat}_2$ are (Mahonian) permutation statistics.

Mahonian d-functions

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- Let Π denote the set of vincular patterns of length at most d. A *d-function* is a statistic of the form

$$\mathsf{stat} = \sum_{\pi \in \Pi} \alpha_\pi (\pi),$$

where $\alpha_\pi \in \mathbb{N}$ and (π) counts occurrences of the pattern π.

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- Let Π denote the set of vincular patterns of length at most *d*. A *d-function* is a statistic of the form

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- Theorem (Babson-Steingrímsson '00)

  For each $d \geq 0$ there is a finite number of *Mahonian d*-functions.

  - Babson-Steingrímsson classify all Mahonian 3-functions.

Name	Vincular pattern statistic	Original reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson
bast′	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson
bast″	$(\underline{13}2) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger
foze′	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger
foze″	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton
sist′	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton
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- Some motivation:
  - Existing bijections for proving Mahonity usually do not restrict to bijections over $\mathrm{Av}_n(\pi)$. A priori no reason for such equidistributions to exist.
  - $|\mathrm{Av}_n(\pi)| = \frac{1}{n+1}\binom{2n}{n}$ for $\pi \in \mathcal{S}_3$. Get induced equidistributions between statistics on other Catalan structures under appropriate bijections (and vice versa).

- Proposition (A. '17)

  Let $\sigma \in \mathrm{Av}_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then

$$\mathrm{mak}(\sigma) = \mathrm{imaj}(\sigma).$$

  Moreover for any $n \geq 1$,

$$\sum_{\sigma \in \mathrm{Av}_n(\pi)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathrm{Av}_n(\pi^{-1})} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

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  - Remark: By above proposition and a result of Stump we have

$$\sum_{\sigma \in \mathsf{Av}_n(231)} q^{\mathrm{maj}(\sigma)+\mathrm{mak}(\sigma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

    (MacMahon's q-analogue of the Catalan numbers)

- Theorem (Dokos-Dwyer-Johnson-Sagan-Selsor '12)

$$\sum_{\sigma \in \mathrm{Av}_n(231)} q^{\mathrm{inv}(\sigma)} = \tilde{C}_n(q),$$

where

$$\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^k \tilde{C}_k(q) \tilde{C}_{n-k-1}(q)$$

(Carlitz-Riordan's q-analogue of the Catalan numbers)

- Theorem (A. '17)

  For any $n \geq 1$,

$$\sum_{\sigma \in \mathrm{Av}_n(321)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma \in \mathrm{Av}_n(321)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)},$$

$$\sum_{\sigma \in \mathrm{Av}_n(123)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)} = \sum_{\sigma \in \mathrm{Av}_n(123)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)}.$$

  where $\mathrm{DB}(\sigma) = \{\sigma_{i+1} : \sigma_i > \sigma_{i+1}\}$, $\mathrm{DT}(\sigma) = \{\sigma_i : \sigma_i > \sigma_{i+1}\}$ and $\mathrm{AB}(\sigma), \mathrm{AT}(\sigma)$ defined similarly.

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  - Proved via an involution $\phi : \mathrm{Av}_n(321) \to \mathrm{Av}_n(321)$.
    - Let $5612379468 \in \mathrm{Av}_9(321)$

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    - Let 561237948 $\in \mathrm{Av}_9(321)$
    - Red letters are left-to-right maxima and Blue letters are non-left-to-right maxima.

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    - Green letters are descent tops and descent bottoms.

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  - Proved via an involution $\phi : \mathrm{Av}_n(321) \to \mathrm{Av}_n(321)$.
    - Let 561237948 $\in \mathcal{S}_9(321)$
    - The involution preserves the relative order of Green letters (descent pairs) and swaps the role of Red (LRMax) and Blue (non-LRMax) letters.
    - We get 561237948 $\mapsto$ 236189457.

An induced equidistribution

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- The bijection $\phi : \mathrm{Av}_n(321) \to \mathrm{Av}_n(321)$ induces an equidistribution on statistics associated with shortened polyominoes (another Catalan structure).

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- A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying
  1. P and Q begin at the same vertex and end at the same vertex.
  2. P stays weakly above Q and the two paths can share E-steps but not N-steps.

  Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$.

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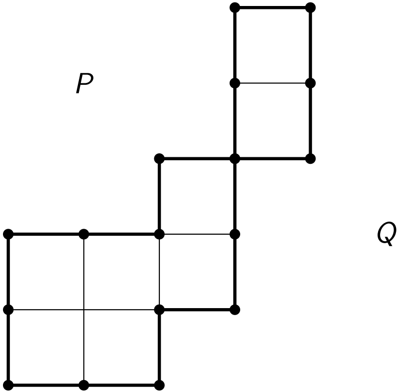
- $|\mathcal{H}_n| = \frac{1}{n+1}\binom{2n}{n}$.

A bijection $\Psi : \mathcal{H}_n \to \mathrm{Av}_n(321)$



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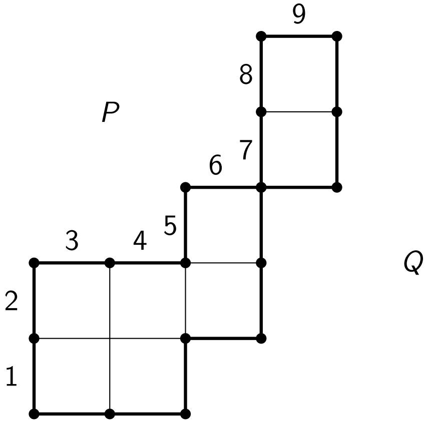

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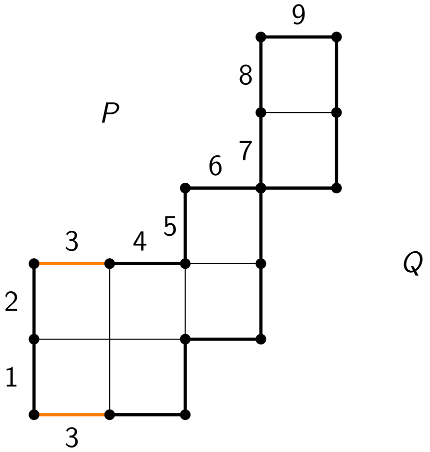

$$\Psi(P, Q) = 3$$

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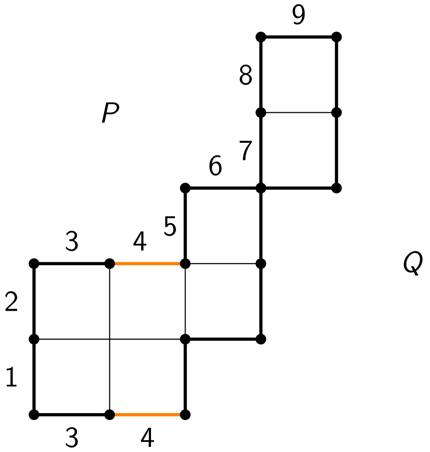
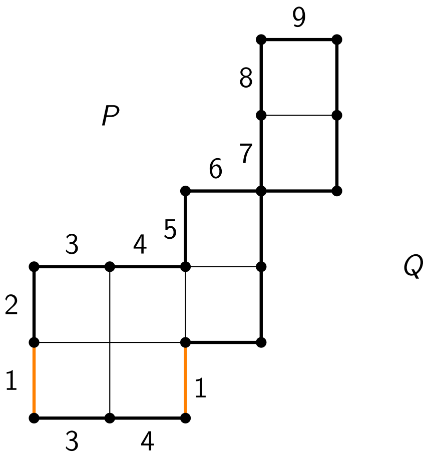


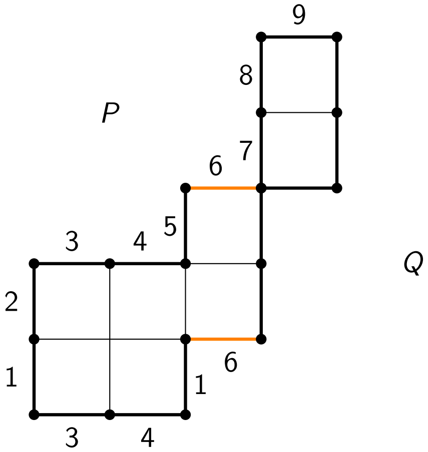
$$\Psi(P, Q) = 34$$

A bijection $\Psi : \mathcal{H}_n \to \mathrm{Av}_n(321)$



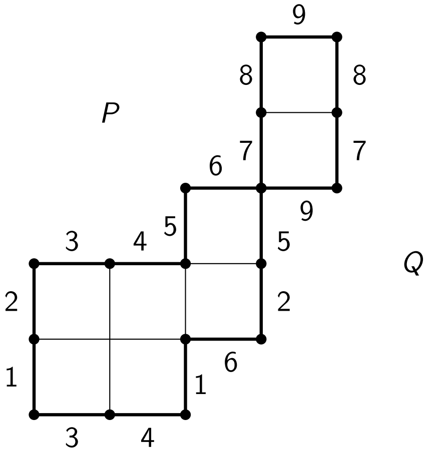
$$\Psi(P, Q) = 341$$

A bijection $\Psi : \mathcal{H}_n \to \mathrm{Av}_n(321)$



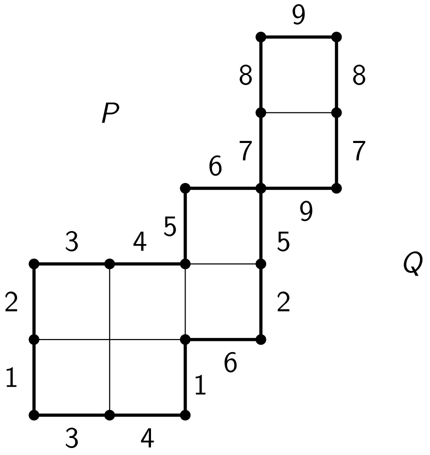
$$\Psi(P, Q) = 3416$$

A bijection $\Psi : \mathcal{H}_n \to \mathrm{Av}_n(321)$



$$\Psi(P, Q) = 341625978$$

A bijection $\Psi : \mathcal{H}_n \to \mathrm{Av}_n(321)$



$$\Psi(P, Q) = 341625978 \in \mathrm{Av}_9(321)$$

Two statistics on polyominoes

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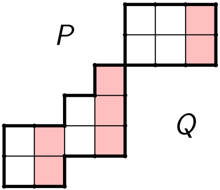
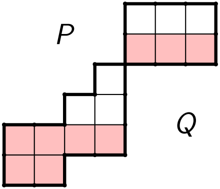
- Let $\mathrm{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$ denote the set of indices of the *valleys* in Q and let $\mathrm{val}(Q) = |\mathrm{Valley}(Q)|$.

Two statistics on polyominoes

- Let $\text{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$ denote the set of indices of the *valleys* in Q and let $\text{val}(Q) = |\text{Valley}(Q)|$.
- Define the statistics *valley-column area*, $\text{vcarea}(P, Q)$, and *valley-row area*, $\text{vrarea}(P, Q)$.



(a) $\text{vcarea}(P, Q) = 2 + 3 + 2 = 7$



(b) $\text{vrarea}(P, Q) = 2 + 4 + 3 = 9$

An induced equidistribution

- Theorem (A. '17)

  For any $n \geq 1$,

$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vcarea}(P,Q)} t^{\mathrm{val}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vrarea}(P,Q)} t^{\mathrm{val}(Q)}.$$

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  - $\mathrm{vcarea}(P,Q) = ((\underline{21}) + (\underline{31}2))\Psi(P,Q)$

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  - $\mathrm{vrarea}(P,Q) = ((\underline{21}) + (2\underline{31}))\Psi(P,Q)$

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  - $\mathrm{vcarea}(P,Q) = ((\underline{21}) + (\underline{31}2))\Psi(P,Q)$
  - $\mathrm{vrarea}(P,Q) = ((\underline{21}) + (2\underline{31}))\Psi(P,Q)$
  - $((\underline{21}) + (\underline{31}2))\phi(\sigma) = ((\underline{21}) + (2\underline{31}))\sigma.$
  - Equidistribution follows via bijection $\Psi^{-1} \circ \phi \circ \Psi$.

Another example

$$\sum_{\sigma \in \mathsf{Av}_n(321)} q^{\mathsf{inv}(\sigma)}$$

Another example

$$\sum_{\sigma \in \mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} =^{(*)} \sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)}$$

(∗) Cheng-Elizalde-Kasraoui-Sagan ’13

$$\mathrm{spea}(P) = \sum_{p \in \mathrm{Peak}(P)} (\mathrm{ht}_P(p) - 1).$$

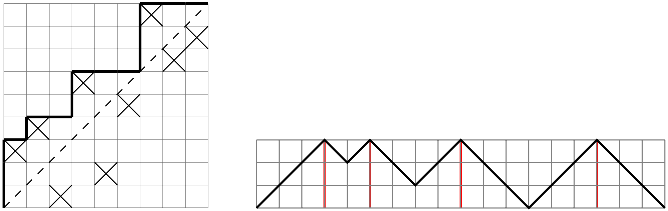

Figure: The Dyck path corresponding to $\sigma = 341625978$ under Krattenthaler’s bijection. Maps inv to spea.

Another example

$$\sum_{\sigma\in \mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{spea}(P)} =^{(*)} \sum_{P\in\mathcal{D}_n} q^{\mathrm{stun}(P)}$$

$(*)$ Cheng-Elizalde-Kasraoui-Sagan '13

$$\mathrm{stun}(P) = \sum_{(i,j)\in \mathrm{Tunnel}(P)} (j-i)/2$$

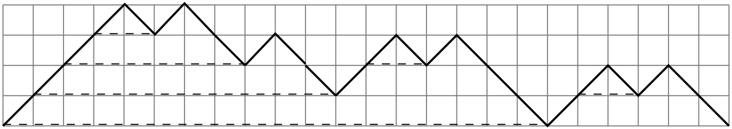

Figure: The tunnel lengths of a Dyck path (indicated with dashes).

Another example

$$\sum_{\sigma \in \mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} =^{(*)} \sum_{P \in \mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)}$$

$(*)$ A. '17

The *mass* corresponding to two consecutive U-steps, is half the number of steps between their matching D-steps (i.e. if $P = UUP'DP''D$, then mass of the pair UU is $|P''|/2$).

$$\begin{aligned} \mathrm{mass}(P) &= \text{ sum of masses over all occurrences of } UU \\ \mathrm{dr}(P) &= \text{ number of double rises in } P. \end{aligned}$$

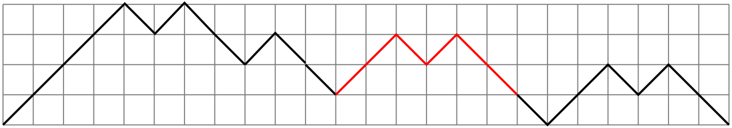

Figure: The mass associated with the first double rise is highlighted in red.

Another example

$$\sum_{\sigma\in\mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)}$$
$$=^{(*)} \sum_{\sigma\in\mathrm{Av}_n(231)} q^{\mathrm{mad}(\sigma)}$$

$(*)$ A. '17

Follows via Knuth's 'standard' bijection

$$f : \mathrm{Av}_n(231) \to \mathcal{D}_n$$
$$213[1,\sigma_1,\sigma_2] \mapsto Uf(\sigma_1)Df(\sigma_2).$$

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$$\sum_{\sigma\in\mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)}$$

$$= \sum_{\sigma\in\mathrm{Av}_n(231)} q^{\mathrm{mad}(\sigma)} =^{(*)} \sum_{\sigma\in\mathrm{Av}_n(132)} q^{\mathrm{inc}(\sigma)}$$

$(*)$ A. '17

$$\mathrm{inc} = \iota_1 + \sum_{k=2}^{\infty}(-1)^{k-1}2^{k-2}\iota_k$$

where $\iota_{k-1} = (12\ldots k)$ is the statistic that counts the number of increasing subsequences of length k in a permutation.

Uses the Catalan continued fraction framework of Brändén-Claesson-Steingrímsson.

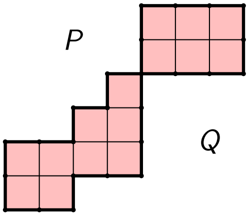
Another example

$$\sum_{\sigma\in \mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)}$$

$$= \sum_{\sigma\in \mathrm{Av}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma\in \mathrm{Av}_n(132)} q^{\mathrm{inc}(\sigma)}$$

Aside

$$\sum_{\sigma\in \mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} \stackrel{(*)}{=} \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{area}(P,Q)}.$$



$(*)$ Cheng-Eu-Fu '07

Another example

$$\sum_{\sigma\in\mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)}$$
$$= \sum_{\sigma\in\mathrm{Av}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma\in\mathrm{Av}_n(132)} q^{\mathrm{inc}(\sigma)}$$

Aside

$$\sum_{\sigma\in\mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{area}(P,Q)}$$
$$\sum_{\sigma\in\mathrm{Av}_n(321)} q^{\mathrm{inv}(\sigma)} =^{(*)} \sum_{P\in\mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

$(*)$ A. '17

$$\mathrm{sups}(P) = \sum_{i\in\mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil.$$

- Several more equidistributions hold between Mahonian 3-functions over $\mathrm{Av}_n(\pi)$!

- Several more equidistributions hold between Mahonian 3-functions over $\mathrm{Av}_n(\pi)$!
- The following table includes all established equidistributions (in black) and all conjectured equidistributions (in red).

	maj	inv	mak	makl	mad	bast	bast′	bast″	foze	foze′	foze″	sist	sist′	sist″
maj	132, 231 213, 312		123, 123 132, 132 132, 312 213, 213 213, 231 231, 132 231, 312 312, 213 312, 231 321, 321	132, 231 213, 312 231, 231 312, 312 321, 321		132, 213 213, 231 231, 213 312, 231	132, 132 231, 132	213, 231 312, 231	132, 132 213, 231 231, 132 312, 231					
inv	•	132, 213 231, 312			231, 312 312, 312 321, 231					231, 132 312, 132 321, 213	231, 132 312, 132 321, 213	231, 213 312, 213 321, 132	231, 231 312, 231 321, 132	231, 132 312, 132 321, 231
mak	•	•	132, 312 213, 231	132, 231 213, 312 231, 312 312, 231 321, 321		132, 213 213, 231 231, 231 312, 213	132, 132 312, 132	213, 231 231, 231	132, 132 213, 231 231, 231 312, 132					
makl	•	•	•			132, 132 231, 213 312, 231	231, 132	312, 231	132, 213 231, 132 312, 231					
mad	•	•	•	•						231, 213 312, 132	231, 213 312, 132	231, 132 312, 213	132, 213 231, 132 312, 231	213, 213 231, 231 312, 132
bast	•	•	•	•	•		213, 132	231, 231	123, 123 213, 132 132, 213 231, 231 312, 312 321, 321					
bast′	•	•	•	•	•	•			132, 132					
bast″	•	•	•	•	•	•	•		231, 231					
foze	•	•	•	•	•	•	•	•						
foze′	•	•	•	•	•	•	•	•	•		132, 132 213, 213	132, 213 213, 132	132, 231 213, 132	132, 132 213, 231
foze″	•	•	•	•	•	•	•	•	•	•		213, 132 132, 213	213, 132 132, 231	132, 132 213, 231
sist	•	•	•	•	•	•	•	•	•	•	•		132, 132 213, 231 312, 312	132, 231 213, 132 231, 312
sist′	•	•	•	•	•	•	•	•	•	•	•	•		132, 231 231, 132
sist″	•	•	•	•	•	•	•	•	•	•	•	•	•	

Thank you for listening!