Equidistributions of Mahonian statistics over pattern avoiding permutations

Nima Amini (KTH, Stockholm)

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Permutation Patterns, Reykjavik
Overall Point
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- Combine the theory of pattern avoidance with the theory of statistics.
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- Study the generating function

\[ F(q) = \sum_{\sigma \in Av_n(\Pi)} q^{\text{stat}(\sigma)}. \]
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- Combine the theory of pattern avoidance with the theory of statistics.
- Study the generating function

\[ F(q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{stat}(\sigma)}. \]

- Example questions: Equidistribution? Recurrence relation? Unimodality/Log-concavity/real-rootedness? etc..
Our focus
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- We study equidistributions of the form

\[ \sum_{\sigma \in \text{Av}_n(\pi_1)} q^{\text{stat}_1(\sigma)} = \sum_{\sigma \in \text{Av}_n(\pi_2)} q^{\text{stat}_2(\sigma)} \]

where \( \pi_1, \pi_2 \in S_3 \) are (classical) patterns and \( \text{stat}_1, \text{stat}_2 \) are (Mahonian) permutation statistics.
Mahonian $d$-functions

$\Pi$ denote the set of vincular patterns of length at most $d$. A $d$-function is a statistic of the form $\text{stat} = \sum_{\pi \in \Pi} \alpha_{\pi}(\pi)$, where $\alpha_{\pi} \in \mathbb{N}$ and $\pi$ counts occurrences of the pattern $\pi$.

Theorem (Babson-Steingrímsson '00) For each $d \geq 0$ there is a finite number of Mahonian $d$-functions.

Babson-Steingrímsson classify all Mahonian 3-functions.
Mahonian d-functions

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- **Theorem (Babson-Steingrímsson '00)**
  For each $d \geq 0$ there is a finite number of Mahonian $d$-functions.
  - Babson-Steingrímsson classify all Mahonian 3-functions.
<table>
<thead>
<tr>
<th>Name</th>
<th>Vincular pattern statistic</th>
<th>Original reference</th>
</tr>
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<tbody>
<tr>
<td>maj</td>
<td>$(132) + (231) + (321) + (21)$</td>
<td>MacMahon</td>
</tr>
<tr>
<td>inv</td>
<td>$(231) + (312) + (321) + (21)$</td>
<td>MacMahon</td>
</tr>
<tr>
<td>mak</td>
<td>$(132) + (312) + (321) + (21)$</td>
<td>Foata-Zeilberger</td>
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<td>makl</td>
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<td>Clarke-Steingrígsson-Zeng</td>
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<tr>
<td>mad</td>
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<tr>
<td>bast</td>
<td>$(132) + (213) + (321) + (21)$</td>
<td>Babson-Steingrígsson</td>
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<tr>
<td>bast′</td>
<td>$(132) + (312) + (321) + (21)$</td>
<td>Babson-Steingrígsson</td>
</tr>
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<tr>
<td>foze</td>
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<tr>
<td>sist</td>
<td>$(132) + (132) + (213) + (21)$</td>
<td>Simion-Stanton</td>
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</table>

Table: Mahonian 3-functions.
Question

Some motivation:

Existing bijections for proving Mahonity usually do not restrict to bijections over $\text{Av}_n(\pi)$. A priori no reason for such equidistributions to exist.

$|\text{Av}_n(\pi)| = \frac{1}{n+1} \left(\frac{2}{n}\right)^n$ for $\pi \in S_3$. Get induced equidistributions between statistics on other Catalan structures under appropriate bijections (and vice versa).
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- Some motivation:
  - Existing bijections for proving Mahonity usually do not restrict to bijections over $\text{Av}_n(\pi)$. A priori no reason for such equidistributions to exist.
  - $|\text{Av}_n(\pi)| = \frac{1}{n+1} \binom{2n}{n}$ for $\pi \in S_3$. Get induced equidistributions between statistics on other Catalan structures under appropriate bijections (and vice versa).
Proposition (A. ’17)

Let $\sigma \in \Av_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then

$$\mak(\sigma) = \imaj(\sigma).$$

Moreover for any $n \geq 1$,

$$\sum_{\sigma \in \Av_n(\pi)} q^{\maj(\sigma)} t^{\des(\sigma)} = \sum_{\sigma \in \Av_n(\pi^{-1})} q^{\mak(\sigma)} t^{\des(\sigma)}.$$
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Remark: By above proposition and a result of Stump we have

$$\sum_{\sigma \in \text{Av}_n(231)} q^{\text{maj}(\sigma)+\text{mak}(\sigma)} = \frac{1}{[n+1]_q} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q$$

(MacMahon’s q-analogue of the Catalan numbers)
Theorem (Dokos-Dwyer-Johnson-Sagan-Selsor ’12)

\[
\sum_{\sigma \in \text{Av}_n(231)} q^{\text{inv}(\sigma)} = \tilde{C}_n(q),
\]

where

\[
\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^k \tilde{C}_k(q) \tilde{C}_{n-k-1}(q)
\]

(Carlitz-Riordan’s q-analogue of the Catalan numbers)
Theorem (A. ’17)

For any \( n \geq 1 \),

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{maj}(\sigma)} x^{\text{DB}(\sigma)} y^{\text{DT}(\sigma)} = \sum_{\sigma \in \text{Av}_n(321)} q^{\text{mak}(\sigma)} x^{\text{DB}(\sigma)} y^{\text{DT}(\sigma)},
\]

\[
\sum_{\sigma \in \text{Av}_n(123)} q^{\text{maj}(\sigma)} x^{\text{AB}(\sigma)} y^{\text{AT}(\sigma)} = \sum_{\sigma \in \text{Av}_n(123)} q^{\text{mak}(\sigma)} x^{\text{AB}(\sigma)} y^{\text{AT}(\sigma)}.
\]

where \( \text{DB}(\sigma) = \{ \sigma_{i+1} : \sigma_i > \sigma_{i+1} \} \), \( \text{DT}(\sigma) = \{ \sigma_i : \sigma_i > \sigma_{i+1} \} \) and \( \text{AB}(\sigma), \text{AT}(\sigma) \) defined similarly.
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**Proved via an involution $\phi : \text{Av}_n(321) \rightarrow \text{Av}_n(321)$.

**Let 5612379468 $\in \text{Av}_9(321)$**
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Let \( 561237948 \in \text{Av}_9(321) \)

- Red letters are left-to-right maxima and Blue letters are non-left-to-right maxima.
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Let $561237948 \in S_9(321)$

Green letters are descent tops and descent bottoms.
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Proved via an involution \( \phi : \text{Av}_n(321) \to \text{Av}_n(321) \).

- Let \( 561237948 \in S_9(321) \)
- The involution preserves the relative order of Green letters (descent pairs) and swaps the role of Red (LRMax) and Blue (non-LRMax) letters.
- We get \( 561237948 \mapsto 236189457 \).
An induced equidistribution

The bijection $\varphi : \text{Av}_n(321) \rightarrow \text{Av}_n(321)$ induces an equidistribution on statistics associated with shortened polyominoes (another Catalan structure).

A shortened polyomino is a pair $(P, Q)$ of north, east lattice paths $P = (P_i)_{n_i=1}$ and $Q = (Q_i)_{n_i=1}$ satisfying

1. $P$ and $Q$ begin at the same vertex and end at the same vertex.
2. $P$ stays weakly above $Q$ and the two paths can share $E$-steps but not $N$-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $H_n$.

$|H_n| = n^{n+1}(2n^n)$. 

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A bijection $\Psi : \mathcal{H}_n \to \text{Av}_n(321)$
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$\Psi(P, Q) = 3$
A bijection $\Psi : \mathcal{H}_n \to \text{Av}_n(321)$

$\Psi(P, Q) = 34$
A bijection $\Psi : \mathcal{H}_n \rightarrow \text{Av}_n(321)$

$\Psi(P, Q) = 341$
A bijection $\Psi : \mathcal{H}_n \to \text{Av}_n(321)$

$\Psi(P, Q) = 3416$
A bijection $\Psi : \mathcal{H}_n \rightarrow \text{Av}_n(321)$

$\Psi(P, Q) = 341625978$
A bijection $\Psi : \mathcal{H}_n \rightarrow \text{Av}_n(321)$

$\Psi(P, Q) = 341625978 \in \text{Av}_9(321)$
Let $\text{Valley}(Q) = \{i : Q_i Q_{i+1} = \text{EN}\}$ denote the set of indices of the valleys in $Q$ and let $\text{val}(Q) = |\text{Valley}(Q)|$.

Define the statistics $\text{valley-column area}$, $\text{vcarea}(P, Q)$, and $\text{valley-row area}$, $\text{vrarea}(P, Q)$.

(a) $\text{vcarea}(P, Q) = 2 + 3 + 2 = 7$

(b) $\text{vrarea}(P, Q) = 2 + 4 + 3 = 9$
Two statistics on polyominoes

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An induced equidistribution

Theorem (A. ’17)
For any $n \geq 1$,

$$
\sum_{(P, Q) \in \mathcal{H}_n} q^{\text{vcarea}(P, Q)} t^{\text{val}(Q)} = \sum_{(P, Q) \in \mathcal{H}_n} q^{\text{vrarea}(P, Q)} t^{\text{val}(Q)}.
$$
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\]

- \( \text{vcarea}(P, Q) = ((21) + (312)) \Psi(P, Q) \)
An induced equidistribution

Theorem (A. ’17)
For any \( n \geq 1 \),

\[
\sum_{(P,Q) \in \mathcal{H}_n} q^{v\text{carea}(P,Q)} t^{\text{val}(Q)} = \sum_{(P,Q) \in \mathcal{H}_n} q^{v\text{rarea}(P,Q)} t^{\text{val}(Q)}.
\]

- \( v\text{carea}(P, Q) = ((21) + (312))\Psi(P, Q) \)
- \( v\text{rarea}(P, Q) = ((21) + (231))\Psi(P, Q) \)
An induced equidistribution

▶ Theorem (A. ’17)

For any $n \geq 1$,

$$\sum_{(P,Q) \in \mathcal{H}_n} q^\text{vcarea}(P,Q) t^\text{val}(Q) = \sum_{(P,Q) \in \mathcal{H}_n} q^\text{vrarea}(P,Q) t^\text{val}(Q).$$

▶ $\text{vcarea}(P, Q) = ((21) + (312))\Psi(P, Q)$
▶ $\text{vrarea}(P, Q) = ((21) + (231))\Psi(P, Q)$
▶ $((21) + (312))\phi(\sigma) = ((21) + (231))\sigma$.
▶ Equidistribution follows via bijection $\Psi^{-1} \circ \phi \circ \Psi$. 
Another example

$$\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)}$$
Another example

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = (\ast) \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)}
\]

(\ast) Cheng-Elizalde-Kasraoui-Sagan '13

\[
\text{spea}(P) = \sum_{p \in \text{Peak}(P)} (\text{ht}_P(p) - 1).
\]

Figure: The Dyck path corresponding to \(\sigma = 341625978\) under Krattenthaler's bijection. Maps inv to spea.
Another example

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)} = \ast \sum_{P \in \mathcal{D}_n} q^{\text{stun}(P)}
\]

\ast \text{ Cheng-Elizalde-Kasraoui-Sagan '13}

\[
\text{stun}(P) = \sum_{(i,j) \in \text{Tunnel}(P)} (j - i)/2
\]

**Figure:** The tunnel lengths of a Dyck path (indicated with dashes).
Another example

\[ \sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{stun}(P)} = (*) \sum_{P \in \mathcal{D}_n} q^{\text{mass}(P)+\text{dr}(P)} \]

(*) A. '17

The mass corresponding to two consecutive $U$-steps, is half the number of steps between their matching $D$-steps (i.e. if $P = UUP'DP''D$, then mass of the pair $UU$ is $|P''|/2$).

\[
\text{mass}(P) = \text{sum of masses over all occurrences of } UU
\]
\[
\text{dr}(P) = \text{number of double rises in } P.
\]

Figure: The mass associated with the first double rise is highlighted in red.
Another example

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{mass}(P)+\text{dr}(P)}
\]

\[= (\ast) \sum_{\sigma \in \text{Av}_n(231)} q^{\text{mad}(\sigma)} \]

\((\ast) \quad \text{A. '17}\)

Follows via Knuth’s ‘standard’ bijection

\[f : \text{Av}_n(231) \rightarrow \mathcal{D}_n\]

\[213[1, \sigma_1, \sigma_2] \mapsto Uf(\sigma_1)Df(\sigma_2).\]
Another example

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}}(\sigma) = \sum_{P \in \mathcal{D}_n} q^{\text{spea}}(P) = \sum_{P \in \mathcal{D}_n} q^{\text{stun}}(P) = \sum_{P \in \mathcal{D}_n} q^{\text{mass}}(P) + dr(P)
\]

\[
= \sum_{\sigma \in \text{Av}_n(231)} q^{\text{mad}}(\sigma) = (*) \sum_{\sigma \in \text{Av}_n(132)} q^{\text{inc}}(\sigma)
\]

\[\text{inc} = \iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k\]

where \(\iota_{k-1} = (12 \ldots k)\) is the statistic that counts the number of increasing subsequences of length \(k\) in a permutation.

Uses the Catalan continued fraction framework of Brändén-Claesson-Steingrímsson.
Another example

\[ \sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}}(\sigma) = \sum_{P \in \mathcal{D}_n} q^{\text{spea}}(P) = \sum_{P \in \mathcal{D}_n} q^{\text{stun}}(P) = \sum_{P \in \mathcal{D}_n} q^{\text{mass}}(P) + dr(P) \]

\[ = \sum_{\sigma \in \text{Av}_n(231)} q^{\text{mad}}(\sigma) = \sum_{\sigma \in \text{Av}_n(132)} q^{\text{inc}}(\sigma) \]

Aside

\[ \sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}}(\sigma) = (*) \sum_{(P,Q) \in \mathcal{H}_n} q^{\text{area}}(P,Q). \]

\[ (*) \text{ Cheng-Eu-Fu '07} \]
Another example

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\text{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\text{mass}(P) + \text{dr}(P)} \\
= \sum_{\sigma \in \text{Av}_n(231)} q^{\text{mad}(\sigma)} = \sum_{\sigma \in \text{Av}_n(132)} q^{\text{inc}(\sigma)}
\]

Aside

\[
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = \sum_{(P,Q) \in \mathcal{H}_n} q^{\text{area}(P,Q)} \\
\sum_{\sigma \in \text{Av}_n(321)} q^{\text{inv}(\sigma)} = (\ast) \sum_{P \in \mathcal{D}_n} q^{\text{sups}(P)}.
\]

(\ast) A. '17

\[
sups(P) = \sum_{i \in \text{Up}(P)} \left\lceil \text{ht}_P(i)/2 \right\rceil.
\]
Several more equidistributions hold between Mahonian 3-functions over $\text{Av}_n(\pi)!$
Several more equidistributions hold between Mahonian 3-functions over $\text{Av}_n(\pi)!$

The following table includes all established equidistributions (in black) and all conjectured equidistributions (in red).
<table>
<thead>
<tr>
<th>maj</th>
<th>inv</th>
<th>mak</th>
<th>makl</th>
<th>mad</th>
<th>bast</th>
<th>bast'</th>
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<th>foze</th>
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<th>foze''</th>
<th>sist</th>
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</tr>
</thead>
<tbody>
<tr>
<td>132, 231</td>
<td><strong>•</strong></td>
<td>231, 312</td>
<td>132, 231</td>
<td><strong>•</strong></td>
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<td>213, 312</td>
<td>132, 231</td>
<td>213, 312</td>
<td></td>
</tr>
</tbody>
</table>

Table: Established equidistributions in black and conjectured equidistributions in red.

For example,
- **•** indicates an established equidistribution.
- Other numbers indicate conjectured equidistributions.
Thank you for listening!