Rare and Not-So-Rare Regions of Random Pattern-Avoiding Permutations

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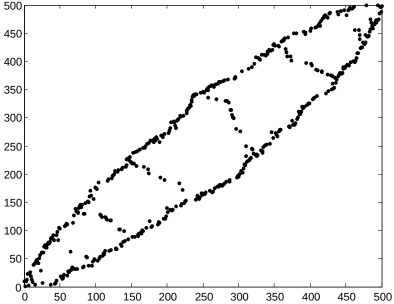
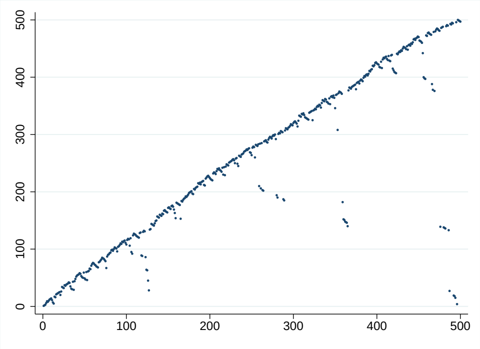
Permutation Patterns 2017

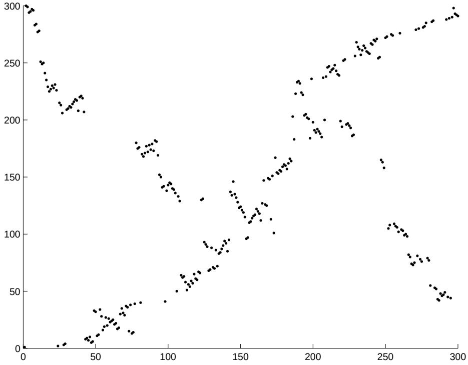
A survey of work with Mahshid Atapour, Lerna Pehlivan, Gökhan Yıldırım, and Yosef Bisk, Hailong Liu, Victor Tsetsulin.



A random 4231-avoiding permutation of length 500



A random 312-avoiding permutation of length 500



A random 2413-avoiding permutation of length 300

Some notation and conventions:

$S_N(\tau)$ is the set of permutations of length N that avoid the pattern τ, where $\tau \in S_k$.

Growth rate: the Stanley-Wilf limit:

$$L(\tau) \;=\; \lim_{N\to\infty} |S_N(\tau)|^{1/N}.$$

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Let P_N^τ denote the uniform probability distribution on $S_N(\tau)$:

$$P_N^\tau(A) \ = \ \frac{|A|}{|S_N(\tau)|} \quad \text{for } A \subset S_N(\tau).$$

We'll consider the plot of a random $\sigma \in S_N(\tau)$ scaled down by N, to the unit square.

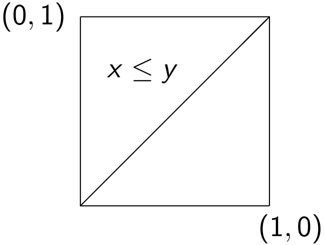
We say that the open set $B \subset \mathbb{R}^2$ is τ-rare if

$$\limsup_{N\to\infty} P_N^\tau \left(\left\{ \left(\frac{i}{N}, \frac{\sigma_i}{N} \right) : i = 1, \ldots, N \right\} \cap B \neq \emptyset \right)^{1/N} \; < \; 1 \, ;$$

i.e., if there exists a $c > 0$ such that, for all sufficiently large N,

$$P_N^\tau \left(\left\{ \left(\frac{i}{N}, \frac{\sigma_i}{N} \right) : i = 1, \ldots, N \right\} \cap B \neq \emptyset \right) \; \leq \; e^{-cN}.$$

Theorem 1 (Atapour & Madras, 2014): *Assume* $\tau \in \mathcal{S}_k$ *with* $\tau_1 > \tau_k$.
(a) Assume $\tau_1 = k$. *Then there is an open* τ*-rare set containing the point* $(0, 1)$.
(b) Assume $\tau_1 < k$. *Then for every open* τ*-rare set* B*, we have* $B \cap \{(x, y) : 0 \leq x \leq y \leq 1\} = \emptyset$.



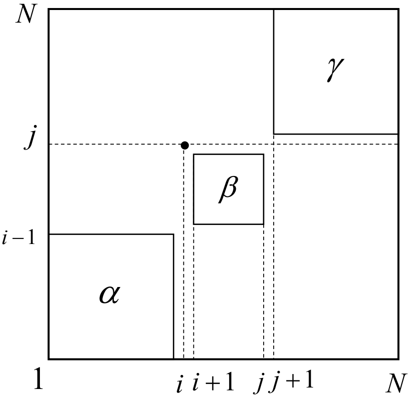
E.g.:

(a): $\tau = 4231$, 312

(b): $\tau = 231$ (rotation of 312),
3142

Proof of Theorem 1(b) [not hard]
Assume $\tau_1 < k$ (and $\tau_1 > \tau_k$), e.g. $\tau = 3142$.
We'll show that for any integer sequences $i(N)$ and $j(N)$ with $1 \leq i(N) \leq j(N) \leq N$,

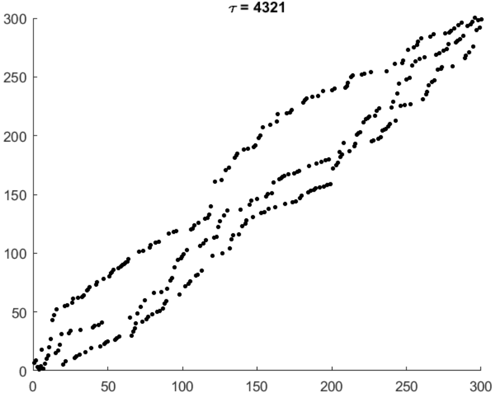
$$\lim_{N\to\infty} P_N^{\tau}\{\, \sigma_{i(N)} \,=\, j(N)\,\}^{1/N} \,=\, 1.$$



$(\tau_k < \tau_1 < k)$

$$P_N^{\tau}\{\,\sigma_i = j\,\} \;\geq\; \frac{|\mathcal{S}_{i-1}(\tau)|\,|\mathcal{S}_{j-i}(\tau)|\,|\mathcal{S}_{N-j}(\tau)|}{|\mathcal{S}_N(\tau)|}$$

Monotone patterns



A random 4321-avoiding permutation of length 300

Theorem 2 (Miner & Pak, 2014; Atapour & Madras, 2014):
Assume

$$\lim_{N\to\infty} \frac{I_N}{N} = \gamma \quad \textit{and} \quad \lim_{N\to\infty} \frac{J_N}{N} = \delta$$

where $0 < \gamma < \delta < 1$. *Then*

$$\begin{aligned} \lim_{N\to\infty} P_N^{321}(\sigma_{I_N} = J_N)^{1/N} &= \frac{1}{4}\, G(\gamma,\delta;1) \; < \; 1 \\ &= \lim_{N\to\infty} P_N^{312}(\sigma_{I_N} = J_N)^{1/N}, \end{aligned}$$

where

$$G(\gamma,\delta;1) \; = \; \frac{(\gamma+\delta)^{\gamma+\delta}(2-\gamma-\delta)^{2-\gamma-\delta}}{\gamma^{\gamma}\,\delta^{\delta}(1-\gamma)^{1-\gamma}(1-\delta)^{1-\delta}}\,.$$

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Theorem 3 (Madras & Pehlivan, 2016 EJC): *Let* $\tau = k(k-1)\cdots 21$. *Assume*

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$$\lim_{N\to\infty} P_N^{\tau}(\sigma_{I_N} = J_N)^{1/N} \;=\; \frac{G(\gamma,\delta;(k-2)^2)}{(k-1)^2} \;<\; 1$$

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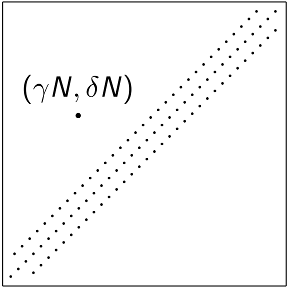
where

$$G(\gamma,\delta;c) \;=\; 4\,c\,g(\gamma,\delta;c)\,g(\delta,\gamma;c)\,g(1-\gamma,1-\delta;c)\,g(1-\delta,1-\gamma;c)$$

$$\textit{and} \qquad g(x,y;c) \;=\; \left(\frac{x(c-1)}{2cx+y-x-\sqrt{(y-x)^2+4cxy}}\right)^{x}.$$

Here, $(k-1)^2 = L(\tau)$ *and* $(k-2)^2 = L((k-1)\cdots 21)$.

$\tau = 4321$:



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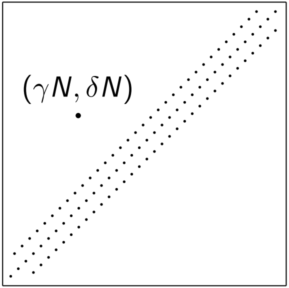
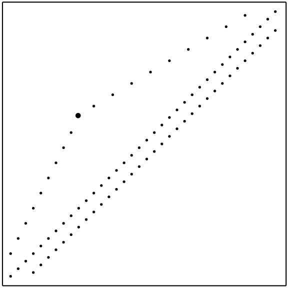


$\Longrightarrow$



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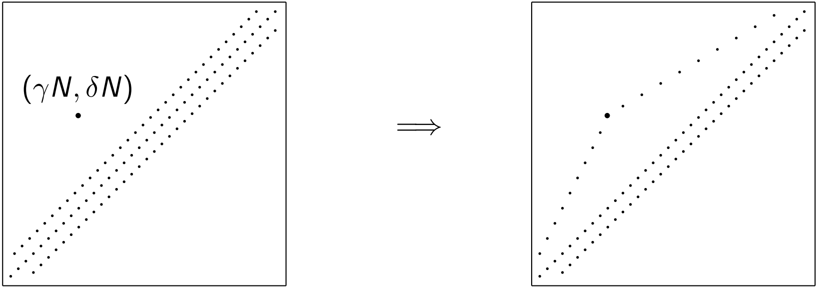

Method also works for $\tau = 21 \ominus \hat{\tau}$: Replace $(k-1)^2$ by $L(\tau)$ and $(k-2)^2$ by $L(1 \ominus \hat{\tau})$ in the theorem.
(correction and comment in preparation)

For a general pattern with $\tau \in S_k$:

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By Theorem 1, $\mathcal{R}^{\uparrow} \neq \emptyset$ if and only if $\tau_1 = k$.

Let $\mathcal{G} = [0,1]^2 \setminus \mathcal{R}$ be the "good region".

Assume $\tau_1 = k$. Let $\mathcal{R}^{\uparrow} = \mathcal{R} \cap \{(x, y) : 0 \leq x \leq y \leq 1\}$.
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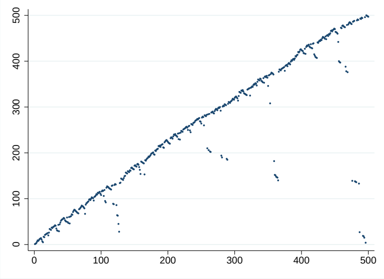
(d) The function $r^{\uparrow}$ has left and right derivatives in $[L(\tau)^{-1}, L(\tau)]$ at every point. (Hence it is strictly increasing and Lipschitz continuous.)

(strengthens results of Atapour & Madras 2014).

Open question: Is $\mathcal{G}$ necessarily convex?

We turn now to regions that are not τ-rare, but sparsely occupied. Primary example: $S_n(312)$ below the diagonal.



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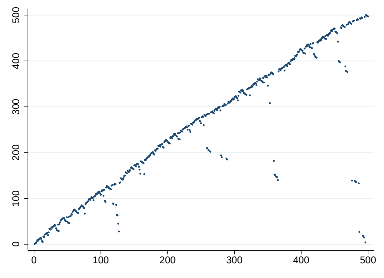

Theorem 5 (Miner & Pak, 2014; Madras & Pehlivan, 2016 RSA)
(a) Assume

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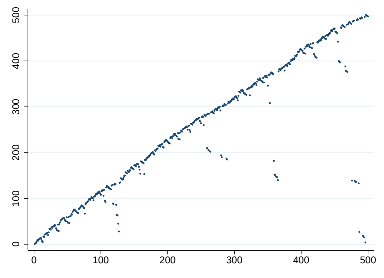

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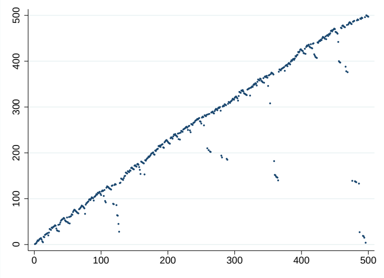
$$\textit{Then} \qquad P_N^{312}(\sigma_{I_N} = J_N) \;\sim\; \frac{N^{-3/2}}{2\sqrt{\pi}\,(1-[\gamma-\delta])^{3/2}\,(\gamma-\delta)^{3/2}}.$$

(b) For fixed $i, j \in \mathbb{N}$:

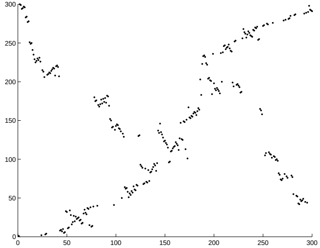
$$\lim_{N\to\infty} P_N^{312}(\sigma_{N+1-i} = j)$$

exists, is nonzero, and can be computed explicitly.
Joint probabilities can also be computed in (a) and (b).



Proofs rely on exact formulas for $P_N^{312}(\sigma_i = j)$.

More challenging pattern: $\tau = 2413$. Recall $\mathcal{R}(2413) = \emptyset$.



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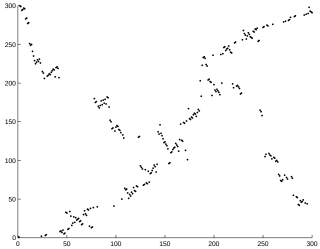
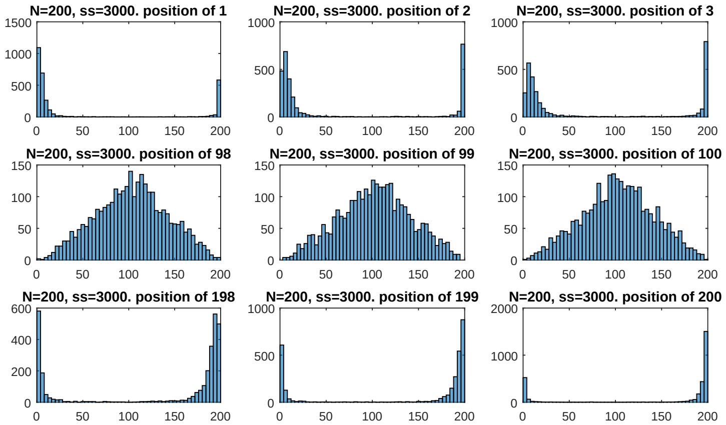


Since $|S_N(2413)| \asymp 8^N N^{-5/2}$, we see from the proof of Theorem 1(b) that $P_N^{2413}(\sigma_i = j)$ cannot decay more slowly that N^{-5}.

$$\left(P_N^{\tau}\{\, \sigma_i \;=\; j \,\} \;\geq\; \frac{|\mathcal{S}_{i-1}(\tau)|\,|\mathcal{S}_{j-i}(\tau)|\,|\mathcal{S}_{N-j}(\tau)|}{|\mathcal{S}_N(\tau)|} \right)$$

Remark: For $i = 1$, we have a lower bound of order $N^{-5/2}$. But so far, we can't say much more than this.

Simulation results from $S_{200}(2413)$ (sample size 3000):



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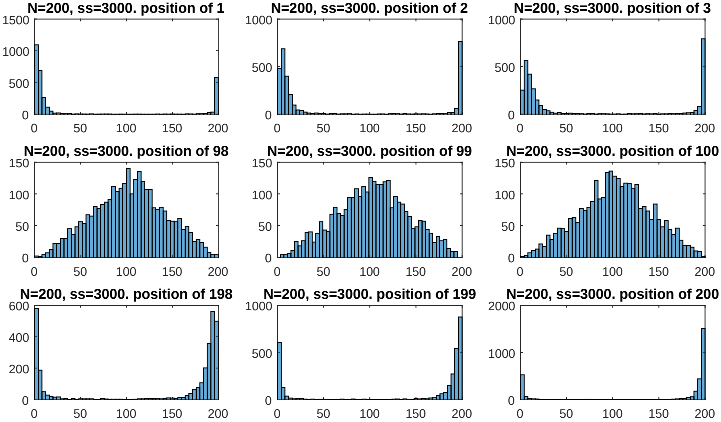
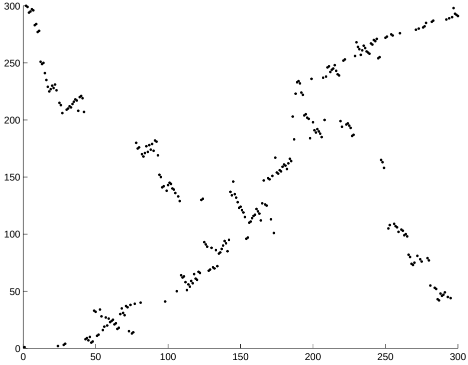


Conjecture: $P_N^{2413}(\sigma_1 = \lfloor N/2 \rfloor) \asymp N^{-2}$.



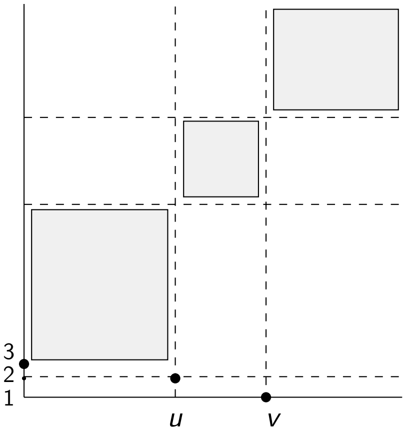
A random 2413-avoiding permutation of length 300

Theorem 6 (Madras & Yıldırım, in preparation):
For any integer $k \geq 3$,

$$\lim_{N\to\infty} P_N^{2413}(\sigma_1 = k) \;=\; 8^{-k} \sum_{m=3}^{k} m\, b_k(m) \left(\frac{32}{27}\right)^{m-1},$$

where $b_k(m)$ is the cardinality of an explicit subset of $S_{k-1}(2413)$.

Exact computation of the right-hand side up to $k = 10$ produces a very good fit to a function of the form C/k^2.



$m = 3$ term for $k = 3$:

$$P_N^{2413}(\sigma_1 = 3) \;=\; \sum_{u<v} \frac{|S_{u-2}(2413)|\,|S_{v-u-1}(2413)|\,|S_{N-v}(2413)|}{|S_N(2413)|}$$

Still lots to explore, theoretically and numerically!

• Does $\mathcal{R}(4231)$ contain everything off the diagonal? If so, are there any patterns whose rare regions have "interesting" shapes?

• Do longer patterns show new structures? Which patterns are likely candidates?

• Can we obtain power laws for probabilities when the rare region is the empty set?