Splittability and 1-Amalgamability of Permutation Classes

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Permutation Patterns
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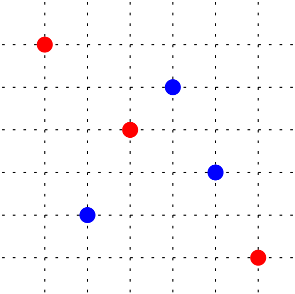
Merges

Definition

Permutation π is a merge of permutations σ and τ if the elements of π can be colored red and blue, so that the red elements are a copy of σ and the blue ones of τ.



One possible merge of 132 and 321 is 624531.

Splittability

Definition

For two sets P and Q of permutations, let $P \odot Q$ be the set of permutations obtained by merging a $\sigma \in P$ with a $\tau \in Q$.

Definition

A permutation class C is splittable if it has two proper subclasses A and B such that $C \subseteq A \odot B$. Otherwise we say that C is unsplittable.

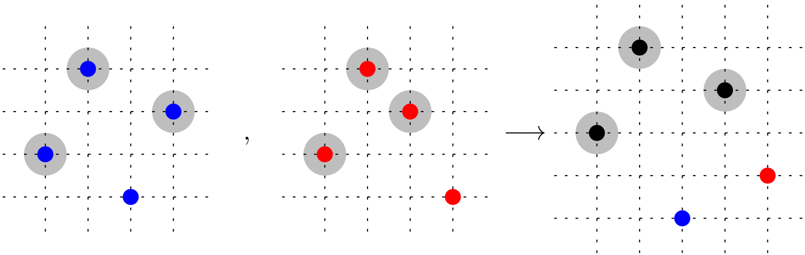
Facts:

- If σ is a simple permutation, then $Av(\sigma)$ is an unsplittable class.
- If σ is a decomposable permutation other than 12, 213 or 132, then $Av(\sigma)$ is a splittable class.

Amalgamation

Definition

Let σ_1 and σ_2 be two permutations, each having a prescribed occurrence of a permutation π. An amalgamation of σ_1 and σ_2 is a permutation obtained from σ_1 and σ_2 by identifying the two prescribed occurrences of π (and possibly identifying some more elements as well).



One possible 132-amalgamation of 2413 and 2431 is the permutation 35142.

Amalgamability

Definition

A permutation class C is

- π-amalgamable if for any two permutations $\sigma_1, \sigma_2 \in C$ and any prescribed occurrences of π in σ_1 and σ_2, there is an amalgamation of σ_1 and σ_2 in C.
- amalgamable if it is π-amalgamable for every $\pi \in C$.
- k-amalgamable if it is π-amalgamable for every $\pi \in C$ of length at most k.

Theorem (Cameron, 2002)

There are only 5 nontrivial amalgamable classes - $Av(12)$, $Av(21)$, $Av(231, 213)$, $Av(132, 213)$ and the class of all permutations.

Fact: If a permutation class C is unsplittable, then C is also 1-amalgamable.

Motivation and plan

Questions

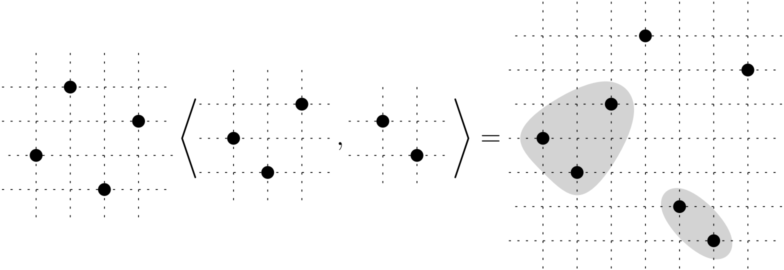
- Is there a splittable 1-amalgamable class?
- Are there infinitely many such classes?

Main result: $Av(1342, 1423)$ is both splittable and 1-amalgamable.

LR-inflations

Definition

For permutation π with k left-to-right minima and $\sigma_1, \ldots, \sigma_k$ non-empty permutations, the LR-inflation of π by the sequence $\sigma_1, \ldots, \sigma_k$ is the inflation of LR-minima of π by $\sigma_1, \ldots, \sigma_k$.



An example of LR-inflation: $2413\langle 213, 21\rangle = 4357216$.

LR-closures

Definition

A permutation class C is closed under LR-inflations if for every $\pi \in C$ and for every k-tuple $\sigma_1, \ldots, \sigma_k$ of permutations from C, the LR-inflation $\pi\langle\sigma_1, \ldots, \sigma_k\rangle$ belongs to C. The closure of C under LR-inflations, denoted C^{LR}, is the smallest class which contains C and is closed under LR-inflations.

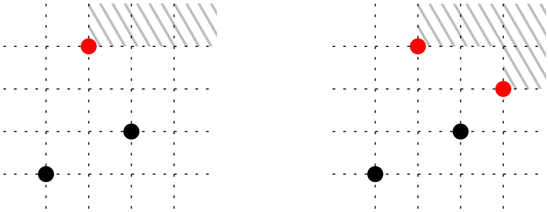
Our plan:

- Show that $Av(1342, 1423)$ is in fact the LR-closure of $Av(123)$.
- Find properties of a permutation class C that imply splittability and 1-amalgamability of C^{LR}.
- Show that $Av(123)$ has these properties.

$Av(1342, 1423) = Av(123)^{\mathrm{LR}}$

Sketch of proof:

- Any $\pi \in Av(123)^{\mathrm{LR}}$ avoids both 1342 and 1423.
- For $\pi \in Av(1342, 1423)$:
  - Consider the right-to-left maxima of π.
  - π does not contain 132 with only one of the letters mapped to a RL-maximum.

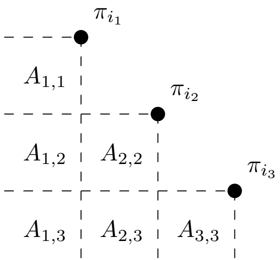


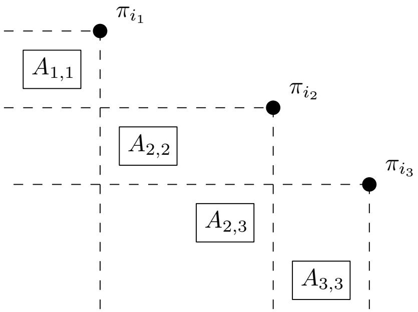
Occurrence of 132 with only the letter 3 mapped to RL-maximum forces 1423.

$Av(1342, 1423) = Av(123)^{\mathrm{LR}}$

Sketch of proof:

- For $\pi \in Av(1342, 1423)$:
  - Split other elements of π into grid defined by the RL-maxima.
  - Show that non-empty sets create a descending sequence of intervals.
  - π is an LR-inflation of 123-avoiding permutation with shorter permutations.

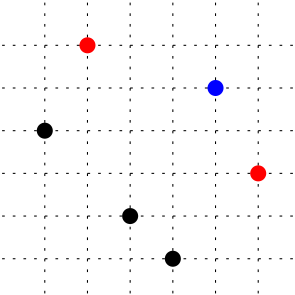




LR-merge

Definition

Permutation π is a LR-merge of permutations σ and τ if the elements of π that are not LR-minima can be colored red and blue, so that the red elements together with LR-minima are a copy of σ and the blue ones of τ.



One possible LR-merge of 45213 and 3214 is 462153.

LR-splittability

Definition

For two sets P and Q of permutations, let $P \odot_{\mathrm{LR}} Q$ be the set of permutations obtained by LR-merging a $\sigma \in P$ with a $\tau \in Q$.

Definition

A permutation class C is LR-splittable if it has two proper subclasses A and B such that $C \subseteq A \odot_{\mathrm{LR}} B$.

Observation: LR-splittability $\Rightarrow$ splittability.

Proposition (Tool #1)

For C, D and E permutation classes, $C \subseteq D \odot_{\mathrm{LR}} E \Rightarrow C^{\mathrm{LR}} \subseteq D^{\mathrm{LR}} \odot_{\mathrm{LR}} E^{\mathrm{LR}}$.

Av(123) and LR-splittability

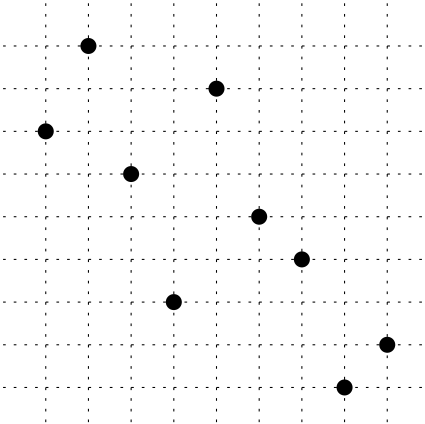
Lemma

$Av(123)$ is LR-splittable.

Constructing a coloring of $\pi \in Av(123)$:

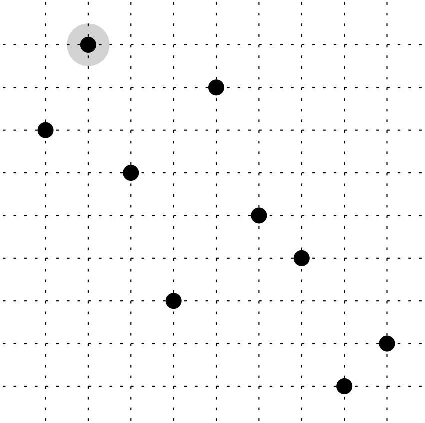
- π is a merge of two descending sequences, LR-minima and the remaining elements.
- We split the non-minimal elements into consecutive runs with a greedy algorithm.
- Finally, every odd run is colored blue and every even run red.

Av(123) and LR-splittability



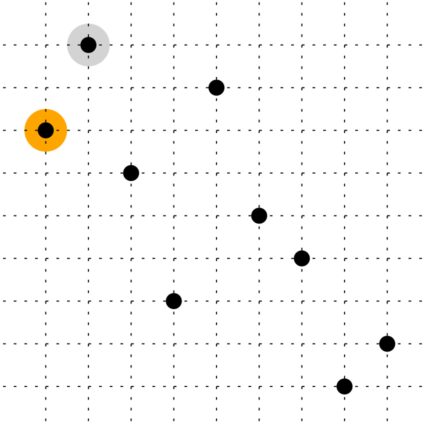
Example of coloring a 123-avoiding permutation.

Av(123) and LR-splittability



Example of coloring a 123-avoiding permutation.

Av(123) and LR-splittability



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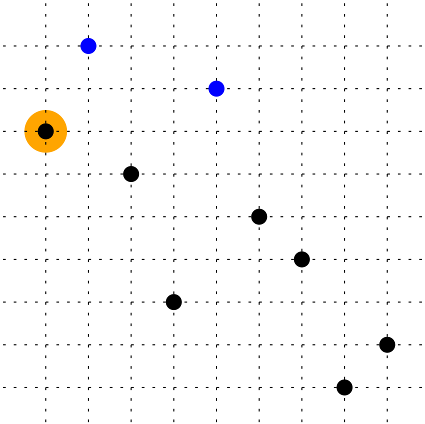

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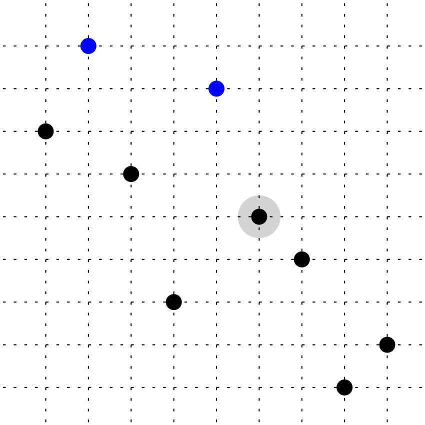

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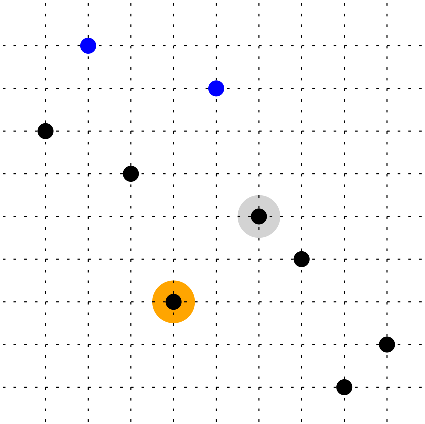

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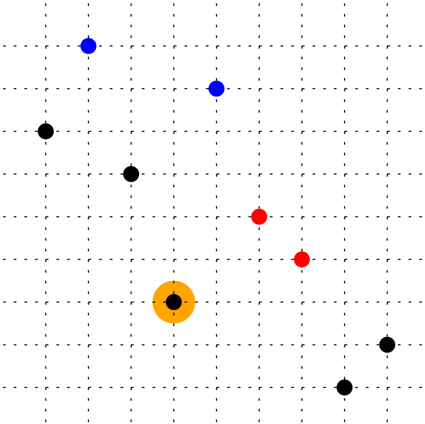

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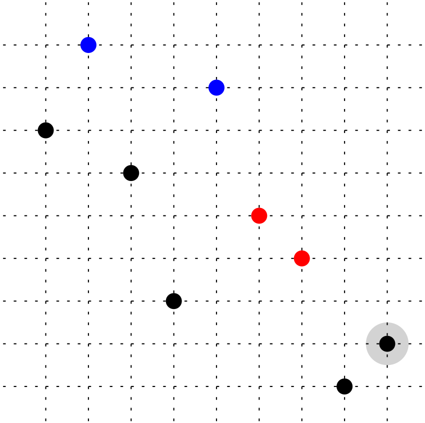

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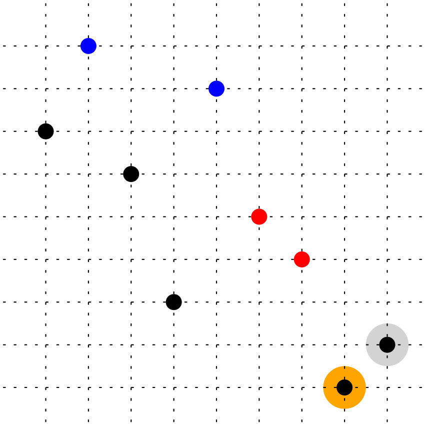

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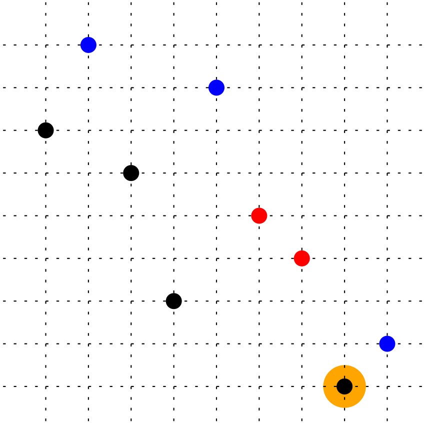
Av(123) and LR-splittability



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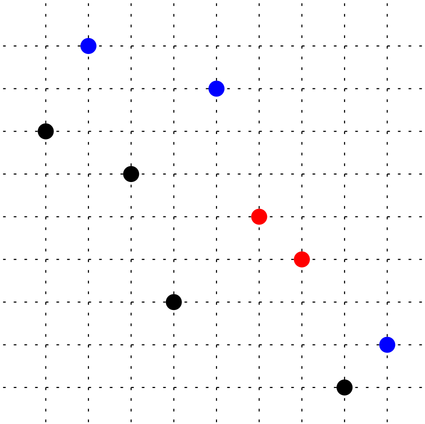
Av(123) and LR-splittability



Example of coloring a 123-avoiding permutation.

Av(123) and LR-splittability

Observation: Two elements from different runs of the same color do not share any LR-minima.

Lemma

$Av(123)$ is LR-splittable.

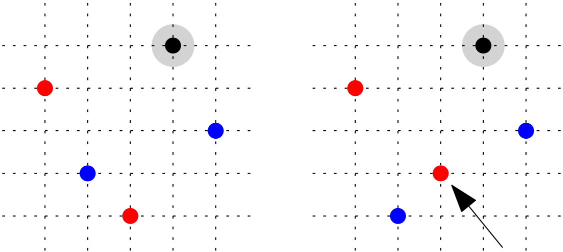
Corollary

$Av(1342, 1423)$ is splittable.

LR-amalgamation

Definition

Let σ_1 and σ_2 be two permutations, each having a prescribed occurrence of a permutation π that does not use any LR-minima. An LR-amalgamation of σ_1 and σ_2 is an amalgamation σ_1 and σ_2 that preserves the property of being a LR-minimum.



Two different 1-amalgamations of 132 and 213, only the left one is a LR-amalgamation.

LR-amalgamability

Definition

A permutation class C is LR-amalgamable if for any two permutations $\sigma_1, \sigma_2 \in C$ and any prescribed occurrence of 1 in σ_1 and σ_2, there is an LR-amalgamation of σ_1 and σ_2 in C.

Proposition (Tool #2)

If a permutation class C is LR-amalgamable then its LR-closure C^{LR} is LR-amalgamable and thus also 1-amalgamable.

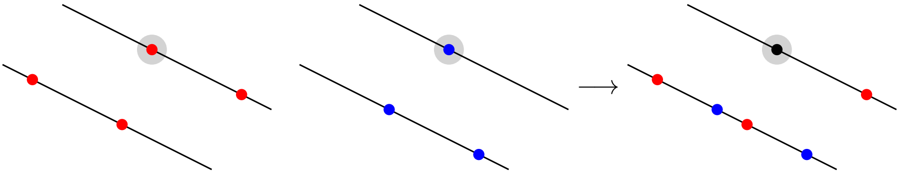
Av(123) and LR-amalgamability

Proposition (Waton, 2007)

The class of permutations that can be drawn on any two parallel lines of negative slope is $Av(123)$.

Lemma

The class $Av(123)$ *is LR-amalgamable.*



Possible LR-amalgamation of 3142 and 231 is the permutation 532614.

Av(1342, 1423)

Corollary

$Av(1342, 1423)$ *is both 1-amalgamable and splittable, which shows that 1-amalgamability* $\not\Rightarrow$ *splittability.*

Further directions

Question

Are there infinitely many 1-amalgamable and splittable classes?

Observation: An element π_i is LR-minimum $\Leftrightarrow$ there is no occurrence of 12 that maps 2 on π_i.

- It is possible to generalize the notions of LR-amalgamability and LR-splittability for elements that are not a specific letter in an occurrence of some permutation σ.
- Maybe that could help find infinitely many 1-amalgamable and splittable classes.

Thank you for your attention!