Splittability and 1-Amalgamability of Permutation Classes

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Permutation Patterns
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Permutation $\pi$ is a merge of permutations $\sigma$ and $\tau$ if the elements of $\pi$ can be colored red and blue, so that the red elements are a copy of $\sigma$ and the blue ones of $\tau$.

One possible merge of 132 and 321 is 624531.
Splittability

**Definition**

For two sets $P$ and $Q$ of permutations, let $P \circ Q$ be the set of permutations obtained by merging a $\sigma \in P$ with a $\tau \in Q$.

**Definition**

A permutation class $C$ is **splittable** if it has two proper subclasses $A$ and $B$ such that $C \subseteq A \circ B$. Otherwise we say that $C$ is **unsplittable**.

**Facts:**

- If $\sigma$ is a simple permutation, then $Av(\sigma)$ is an unsplittable class.
- If $\sigma$ is a decomposable permutation other than 12, 213 or 132, then $Av(\sigma)$ is a splittable class.
**Amalgamation**

**Definition**

Let $\sigma_1$ and $\sigma_2$ be two permutations, each having a prescribed occurrence of a permutation $\pi$. An amalgamation of $\sigma_1$ and $\sigma_2$ is a permutation obtained from $\sigma_1$ and $\sigma_2$ by identifying the two prescribed occurrences of $\pi$ (and possibly identifying some more elements as well).

One possible 132-amalgamation of 2413 and 2431 is the permutation 35142.
Amalgamability

**Definition**

A permutation class $C$ is

- **$\pi$-amalgamable** if for any two permutations $\sigma_1, \sigma_2 \in C$ and any prescribed occurrences of $\pi$ in $\sigma_1$ and $\sigma_2$, there is an amalgamation of $\sigma_1$ and $\sigma_2$ in $C$.
- **amalgamable** if it is $\pi$-amalgamable for every $\pi \in C$.
- **$k$-amalgamable** if it is $\pi$-amalgamable for every $\pi \in C$ of length at most $k$.

**Theorem (Cameron, 2002)**

*There are only 5 nontrivial amalgamable classes - $\text{Av}(12)$, $\text{Av}(21)$, $\text{Av}(231, 213)$, $\text{Av}(132, 213)$ and the class of all permutations.*

**Fact:** If a permutation class $C$ is unsplittable, then $C$ is also 1-amalgamable.
Motivation and plan

Questions

- Is there a splittable 1-amalgamable class?
- Are there infinitely many such classes?

**Main result:** $\text{Av}(1342, 1423)$ is both splittable and 1-amalgamable.
LR-inflations

Definition

For permutation $\pi$ with $k$ left-to-right minima and $\sigma_1, \ldots, \sigma_k$ non-empty permutations, the LR-inflation of $\pi$ by the sequence $\sigma_1, \ldots, \sigma_k$ is the inflation of LR-minima of $\pi$ by $\sigma_1, \ldots, \sigma_k$.

An example of LR-inflation: $2413 \langle 213, 21 \rangle = 4357216$. 
LR-closures

Definition

A permutation class $C$ is closed under LR-inflations if for every $\pi \in C$ and for every $k$-tuple $\sigma_1, \ldots, \sigma_k$ of permutations from $C$, the LR-inflation $\pi \langle \sigma_1, \ldots, \sigma_k \rangle$ belongs to $C$. The closure of $C$ under LR-inflations, denoted $C_{LR}$, is the smallest class which contains $C$ and is closed under LR-inflations.

Our plan:

- Show that $Av(1342, 1423)$ is in fact the LR-closure of $Av(123)$.
- Find properties of a permutation class $C$ that imply splittability and 1-amalgamability of $C_{LR}$.
- Show that $Av(123)$ has these properties.
Sketch of proof:

- Any $\pi \in \text{Av}(123)^{LR}$ avoids both 1342 and 1423.
- For $\pi \in \text{Av}(1342, 1423)$:
  - Consider the right-to-left maxima of $\pi$.
  - $\pi$ does not contain 132 with only one of the letters mapped to a RL-maximum.

Occurrence of 132 with only the letter 3 mapped to RL-maximum forces 1423.
$\text{Av}(1342, 1423) = \text{Av}(123)^{LR}$

Sketch of proof:
- For $\pi \in \text{Av}(1342, 1423)$:
  - Split other elements of $\pi$ into grid defined by the RL-maxima.
  - Show that non-empty sets create a descending sequence of intervals.
  - $\pi$ is an LR-inflation of 123-avoiding permutation with shorter permutations.
Definition

Permutation $\pi$ is a LR-merge of permutations $\sigma$ and $\tau$ if the elements of $\pi$ that are not LR-minima can be colored red and blue, so that the red elements together with LR-minima are a copy of $\sigma$ and the blue ones of $\tau$.

One possible LR-merge of 45213 and 3214 is 462153.
LR-splittability

**Definition**

For two sets $P$ and $Q$ of permutations, let $P \odot_{LR} Q$ be the set of permutations obtained by LR-merging a $\sigma \in P$ with a $\tau \in Q$.

**Definition**

A permutation class $C$ is **LR-splittable** if it has two proper subclasses $A$ and $B$ such that $C \subseteq A \odot_{LR} B$.

**Observation:** LR-splittability $\Rightarrow$ splittability.

**Proposition (Tool #1)**

*For $C$, $D$ and $E$ permutation classes, $C \subseteq D \odot_{LR} E \Rightarrow C^{LR} \subseteq D^{LR} \odot_{LR} E^{LR}$.**
Av(123) and LR-splittability

Lemma

Av(123) is LR-splittable.

Constructing a coloring of $\pi \in \text{Av}(123)$:

- $\pi$ is a merge of two descending sequences, LR-minima and the remaining elements.
- We split the non-minimal elements into consecutive runs with a greedy algorithm.
- Finally, every odd run is colored blue and every even run red.
Example of coloring a 123-avoiding permutation.
Av(123) and LR-splittability

Example of coloring a 123-avoiding permutation.
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Av(123) and LR-splittability

Example of coloring a 123-avoiding permutation.
**Observation:** Two elements from different runs of the same color do not share any LR-minima.

**Lemma**

$\text{Av}(123)$ is LR-splittable.

**Corollary**

$\text{Av}(1342, 1423)$ is splittable.
**Definition**

Let $\sigma_1$ and $\sigma_2$ be two permutations, each having a prescribed occurrence of a permutation $\pi$ that does not use any LR-minima. An **LR-amalgamation** of $\sigma_1$ and $\sigma_2$ is an amalgamation $\sigma_1$ and $\sigma_2$ that preserves the property of being a LR-minimum.

Two different 1-amalgamations of 132 and 213, only the left one is a LR-amalgamation.
LR-amalgamability

Definition
A permutation class $C$ is **LR-amalgamable** if for any two permutations $\sigma_1, \sigma_2 \in C$ and any prescribed occurrence of 1 in $\sigma_1$ and $\sigma_2$, there is an LR-amalgamation of $\sigma_1$ and $\sigma_2$ in $C$.

Proposition (Tool #2)
If a permutation class $C$ is LR-amalgamable then its LR-closure $C^{LR}$ is LR-amalgamable and thus also 1-amalgamable.
**Proposition (Waton, 2007)**

The class of permutations that can be drawn on any two parallel lines of negative slope is $\text{Av}(123)$.

**Lemma**

The class $\text{Av}(123)$ is LR-amalgamable.

Possible LR-amalgamation of 3142 and 231 is the permutation 532614.
Corollary

Av(1342, 1423) is both 1-amalgamable and splittable, which shows that 1-amalgamability $\nRightarrow$ splittability.
Further directions

Question
Are there infinitely many 1-amalgamable and splittable classes?

Observation: An element $\pi_i$ is LR-minimum $\iff$ there is no occurrence of 12 that maps 2 on $\pi_i$.

- It is possible to generalize the notions of LR-amalgamability and LR-splittability for elements that are not a specific letter in an occurrence of some permutation $\sigma$.
- Maybe that could help find infinitely many 1-amalgamable and splittable classes.
Thank you for your attention!