Pattern Avoiding Linear Extensions of Rectangular Posets

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June 26, 2017

Permutation Patterns: Reykjavik, Iceland

Introduction

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To state the most general of these problems, suppose n is a positive integer, $\Uparrow$ is a partial order on $[n]$, and $\sigma_1, \ldots, \sigma_k$ are permutations. Then Yakoubov's problem is to determine how many permutations π of $[n]$ avoid $\sigma_1, \ldots, \sigma_k$ and also have the property that if $\pi(i) \Uparrow \pi(j)$ then $i < j$. In other words, how many linear extensions of the poset $([n], \Uparrow)$ avoid $\sigma_1, \ldots, \sigma_k$?

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Yakoubov obtains simple closed formulas for the number of linear extensions of posets she calls combs which avoid various sets of patterns of length three.

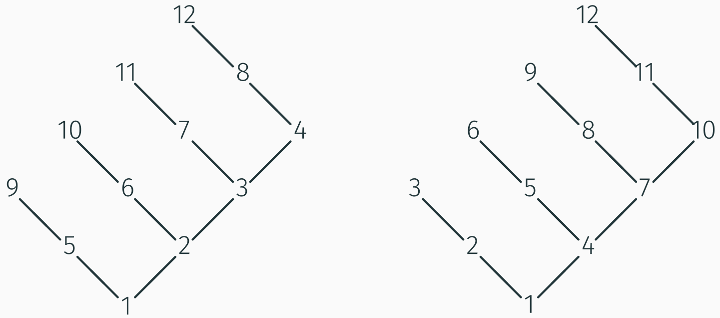

Figure 1: The Hasse diagrams of a comb of type α (left) and β (right).

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- q - analogues
- Some conjectures

Background and Notation

For example, if P is the poset whose Hasse diagram is given in Figure 2, then P has four linear extensions: 25413, 25431, 52413, and 52431.

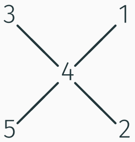


Figure 2: The Hasse diagram of a poset on [5].

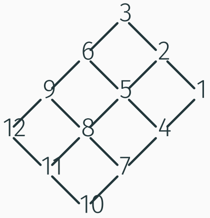


Figure 3: The Hasse diagram of the rectangular poset $EN_{4,3}$.

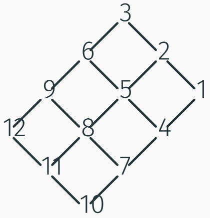
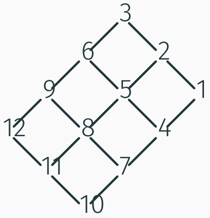


Figure 3: The Hasse diagram of the rectangular poset $EN_{4,3}$.

We write s to denote the number of elements in each spine.



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We write s to denote the number of elements in each spine.
We write t to denote the number of elements in each tooth.

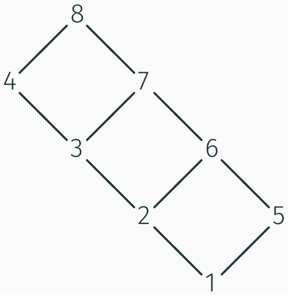

Figure 4: The Hasse diagram of the rectangular poset $SW_{2,4}$.

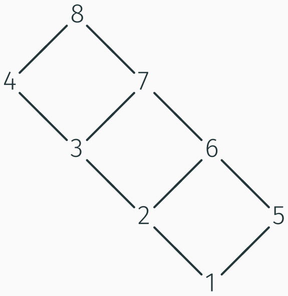

Figure 4: The Hasse diagram of the rectangular poset $SW_{2,4}$.

Good news: we need only concern ourselves with two of them.

As a result of these observations, we see we can restrict our attention to $EN_{s,t}(\sigma_1, \ldots, \sigma_n)$ and $NE_{s,t}(\sigma_1, \ldots, \sigma_n)$.

In addition to reducing the collection of posets we need to consider, we can also reduce the collection of forbidden patterns we need to consider using the standard reverse complement map.

Before we do this, it's natural to ask what happens when there are no patterns to avoid. That is, how many linear extensions of $EN_{s,t}$ and $NE_{s,t}$ are there?

The classical hook length formula for the number of standard tableaux of an arbitrary partition shape gives us the following result.

Proposition

For any positive integers s and t, the number of linear extensions of $EN_{s,t}$ (or $NE_{s,t}$) is $(st)!\prod_{j=1}^{t}\frac{(s+t-j)!}{(j-1)!}$.

Since the Catalan numbers are so ubiquitous, it's worth noting their appearance in a special case of Proposition 1.

Corollary

For any positive integer n, the number of linear extensions of any of $EN_{n,2}$, $NE_{n,2}$ $EN_{2,n}$, or $NE_{2,n}$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Results

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- $|EN_{a,2}(123)| = |EN_{2,a}(321)| = C_a$
- $|NE_{s,t}(213)| = t^{s-1}$
- $|NE_{s,t}(213, 132) = 2^{s-1}$

$|NE_{s,t}(123)|$

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$s \setminus t$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	5	14	42	132
3	1	5	33	234	1706	12618
4	1	14	238	4146	72152	1246804
5	1	42	1782	75187	3099106	
6	1	132	13593	1378668		
7	1	429	104756	25430445		

Table 1: $|NE_{s,t}(123)|$ for small s and t.

Patterns of length 4

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Theorem

For all $s, t \geq 1$,

$$|EN_{s,t}(1243)| = \frac{1}{(t-1)s+1}\binom{st}{s}.$$

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For all $s \geq 0$ and all $t \geq 2$, a *t-Fuss-Catalan path of semilength s* is a lattice path consisting of s unit East and $(t-1)s$ unit North steps, with the property that each initial string of steps includes at least $t-1$ times as many Ns as Es. Equivalently, a t-Fuss-Catalan path must remain on or above the line $y = (t-1)x$.

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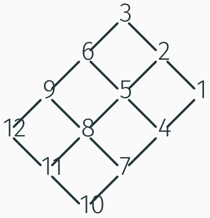
Note that the 2-Fuss-Catalan paths are the classical Catalan paths.

10 11 7 12 8 9 4 5 1 6 2 3



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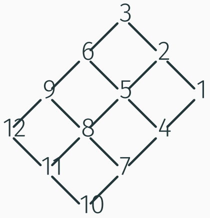

Positions of the roots: $(1, 3, 7, 9)$

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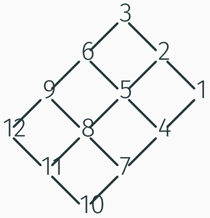

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Complement: $(12, 10, 6, 4)$

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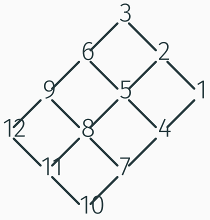

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NNNENENENNNENE

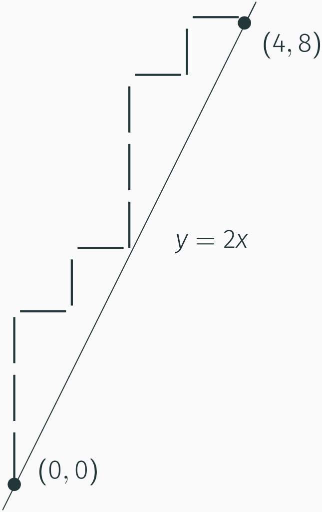

Figure 5: The 4-Fuss-Catalan path *NNNENENENNNENE* and the line $y = 2x$.

q-analogues

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Definition

For any rectangular poset P and any pattern σ, let $P(\sigma)(q)$ be the generating function given by

$$P(\sigma)(q) = \sum_{\pi \in P(\sigma)} q^{\mathrm{inv}(\pi)}.$$

Also let $[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$ for $n \geq 1$.

Theorem

For all $s \geq 1$ *and all* $t \geq 1$,

$$NE_{s,t}(213)(q) = q^{s\binom{t}{2}+t\binom{s}{2}+\frac{(s-1)(t-1)(st-2)}{2}}[t]_q^{s-1}.$$

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For $EN_{s,t}(1243)$, we give the range of possible inversion numbers.

$s \setminus t$	1	2	3	4	5	6
1	0	0	0	0	0	0
2	1	3-4	7-9	13-16	21-25	31-36
3	3	9-12	21-27	39-48	63-75	93-108
4	6	18-24	42-54	78-96	126-150	196-216
5	10	30-40	70-90	130-160	210-250	310-360
6	15	45-60	105-135	195-240	315-375	465-540

Table 2: The range of inversion numbers (minimum-maximum) for linear extensions of $EN_{s,t}(1243)$.

Theorem

For any positive integers s and t, the inversion numbers of the linear extensions of $EN_{s,t}$ which avoid 1243 have minimum $(t^2 - t + 1)\binom{s}{2}$ and maximum $t^2\binom{s}{2}$.

Conjectures

We have found that the following conjecture holds for $t \leq 9$.

Conjecture

For all $t \geq 1$,

$$EN_{3,2t-1}(1243)(q) = q^{3(t^2-t+1)}[2t-1]_q[4t-1]_q.$$

Conjecture

For all $s \geq 1$,

$$EN_{s,3}(2143)(q) = q^{9\binom{s}{2}} f_s\left(\frac{1}{q}\right),$$

where $f_s(q)$ *is defined by* $f_0(q) = 1$, $f_1(q) = 1$, *and* $f_s(q) = (1 + q + 2q^2)f_{s-1}(q) + q^3 f_{s-2}(q)$ *for* $s \geq 2$.

Conjectures about $f_s(q)$

- For all $s \geq 2$, the coefficient of q^3 in $f_s(q)$ is the binomial transform of the sequence obtained by interleaving $n+1$ and $2n+1$. This is OEIS sequence A098156.
- For all $s \geq 2$, the coefficient of q^{2s-2} in $f_s(q)$ the number of compositions of $s+9$ into s parts, none of which is 2 or 3. This is OEIS sequence A134465.
- For all $s \geq 2$, the coefficient of q^{s+2} in $f_s(q)$ is the number of jumps in all binary trees with s edges. This is OEIS sequence A127531.
- For all $s \geq 2$, the coefficient of q^s in $f_s(q)$ is given by OEIS sequence A072547.
- For all $s \geq 2$, the coefficient of q^{s+1} in $f_s(q)$ is $\binom{2s+1}{s-1}$.
- For all $s \geq 2$, the coefficient of q^{s-1} in $f_s(q)$ is the number of hill-free Dyck paths of semilength s. This is OEIS sequence A116914.
- For all $s \geq 0$, the sequence of coefficients of $f_s(q)$ is unimodal.

Other q-analogues

Let $c_n(q)$ denote the polynomial given by

$$c_n(q) = \sum_{\pi \in CW_n} q^{\mathrm{maj}(\pi)},$$

where the sum on the right is over all Catalan words of length $2n$. With this notation we have the following conjecture, which we have verified for $s \leq 9$ and $t \leq 9$.

Conjecture

For all $s \geq 1$ and all $t \geq 1$,

- $EN_{2,t}(321)[q] = q^t c_t(q)$;
- $EN_{s,2}(123)[q] = q^{2\binom{s}{2}} c_s(q)$;
- $NE_{s,2}(123)[q] = q^{s^2} c_s(q)$;
- $NE_{2,t}(123)[q] = q^{t^2} c_t(q)$.

The paper is on the arXiv, and will appear in the Journal of Combinatorics.

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