Sorting with Pop Stacks

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joint work with
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Stack sortable permutations

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Theorem (Knuth, 1973)

$\pi \in \mathcal{S}_n$ is 1-stack sortable iff π avoids 231. There are $C_n = \frac{\binom{2n}{n}}{n+1}$ such permutations.

Stack sortable permutations

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Theorem (Knuth, 1973)

$\pi \in \mathcal{S}_n$ is 1-stack sortable iff π avoids 231. There are $C_n = \frac{\binom{2n}{n}}{n+1}$ such permutations.

Theorem (West, 1990)

$\pi \in \mathcal{S}_n$ is 2-stack sortable iff π avoids 2341 and $3\bar{5}241$.

Theorem (Zeilberger, 1992)

There are $\frac{2(3n)!}{(n+1)!(2n+1)!}$ 2-stack sortable permutations of length n.

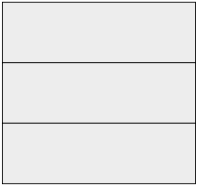
Pop stacks

Pop Stack Operations

A *pop stack* is a last-in first-out data-structure with two operations:

- *push* – remove the first element from input and put it on the top of the stack,
- *pop* – remove all elements from the stack and put them on the end of the output.

Input: 21534



Output: $P(21534) = \ldots$

Pop stacks

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A *pop stack* is a last-in first-out data-structure with two operations:

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Input: 1534

Output: $P(21534) = \ldots$

Pop stacks

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Input: 534

Output: $P(21534) = \ldots$

Pop stacks

Pop Stack Operations

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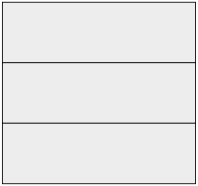
Input: 534



Output: $P(21534) = 12\ldots$

Pop stacks

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- *pop* – remove all elements from the stack and put them on the end of the output.

Input: 34

Output: $P(21534) = 12\ldots$

Pop stacks

Pop Stack Operations

A *pop stack* is a last-in first-out data-structure with two operations:

- *push* – remove the first element from input and put it on the top of the stack,
- *pop* – remove all elements from the stack and put them on the end of the output.

Input: 4

Output: $P(21534) = 12\ldots$

Pop stacks

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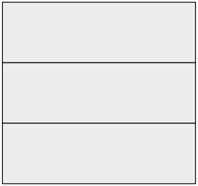
Input: 4



21534 is *not* 1-pop-stack sortable.

Output: $P(21534) = 1235\ldots$

Pop stacks

Pop Stack Operations

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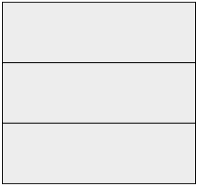
Input:



21534 is *not* 1-pop-stack sortable.

Output: $P(21534) = 12354$

Pop stacks

Pop Stack Operations

A *pop stack* is a last-in first-out data-structure with two operations:

- *push* – remove the first element from input and put it on the top of the stack,
- *pop* – remove all elements from the stack and put them on the end of the output.

In general,
if $\pi_1 \cdots \pi_i$ is the longest decreasing prefix of π,
write $R = \pi_{i+1} \cdots \pi_n$
so that $\pi = \pi_1 \cdots \pi_i R$.

$$P(\pi_1 \cdots \pi_i R) = \pi_i \cdots \pi_1 P(R).$$

Pop-stack sortable permutations

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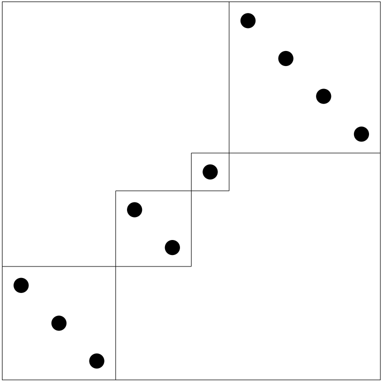
If $P(\pi) = 12\cdots n$, then π is layered.



Theorem (Avis and Newborn, 1981)

$\pi \in \mathcal{S}_n$ is 1-pop-stack sortable iff π avoids 231 and 312.
There are 2^{n-1} such permutations.

Compositions

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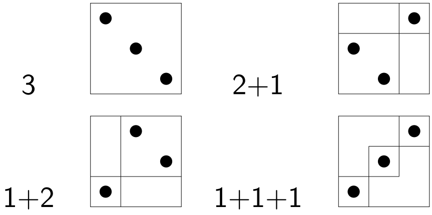
Definition

Composition of n – an ordered arrangement of positive integers whose sum is n

Example: $n = 3$



2-Pop-stack sortability

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Definition

A permutation $\pi \in \mathcal{S}_n$ is 2-pop-stack sortable if and only if $P(P(\pi)) = 1 \cdots n$.

Notation

If $P(P(\pi)) = 1 \cdots n$, write $\pi \in \mathcal{P}_{2,n}$.
Let $\mathcal{P}_2 = \bigcup_{n \geq 1} \mathcal{P}_{2,n}$.

Example: $21534 \in \mathcal{P}_{2,5}$ because

$$P(P(21534)) = P(12354) = 12345.$$

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Example: $21534 \in \mathcal{P}_{2,5}$ because

$$P(P(21534)) = P(12354) = 12345.$$

Definition

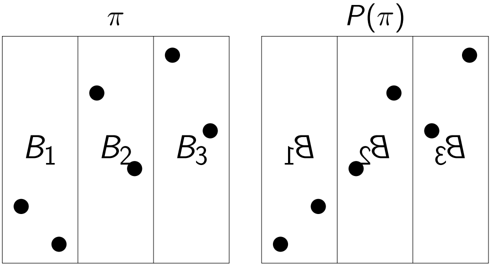
A *block* of a permutation is a maximal contiguous decreasing subsequence.

Example: $\pi = 21534$ has blocks $B_1 = 21$, $B_2 = 53$, $B_3 = 4$.

2-Pop-stack sortability

Claim

π with blocks $B_1, \ldots, B_\ell$ is 2-pop-stack sortable iff either $\max(B_i) < \min(B_{i+1})$ or $\max(B_i) = \min(B_{i+1}) + 1$ for $1 \leq i \leq \ell - 1$.



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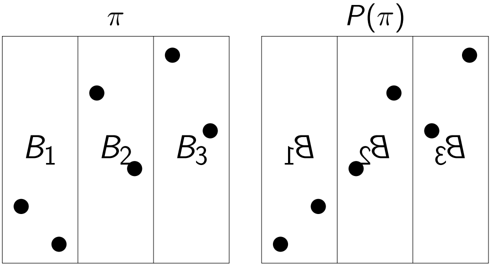

Theorem (P and Smith, 2017+)

π is 2-pop-stack sortable iff π avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

2-Pop-stack sortability

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Lemma

π with blocks $B_1, \ldots, B_\ell$ is 2-pop-stack sortable iff either $\max(B_i) < \min(B_{i+1})$ or $\max(B_i) = \min(B_{i+1}) + 1$ for $1 \leq i \leq \ell - 1$.

- Given a composition c, let $f(c)$ be the number of pairs of adjacent summands that aren't both 1.

Example: $f(1+2+1+1) = 2$.

2-Pop-stack sortability

Lemma

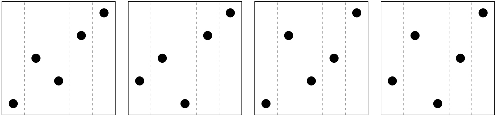
π with blocks $B_1, \ldots, B_\ell$ is 2-pop-stack sortable iff either $\max(B_i) < \min(B_{i+1})$ or $\max(B_i) = \min(B_{i+1}) + 1$ for $1 \leq i \leq \ell - 1$.

- Given a composition c, let $f(c)$ be the number of pairs of adjacent summands that aren't both 1.

  Example: $f(1+2+1+1) = 2$.

- There are $2^{f(c)}$ 2-pop-stack sortable permutations with block structure given by c.

  Example: The 2-pop-stack sortable permutations with block structure $1+2+1+1$ are:



Enumeration via ascents

Notation

- $a(n,k)$ is the number of 2-pop-stack sortable permutations of length n with k ascents.
- $b(n,k)$ is the number of permutations counted by $a(n,k)$ with last block of size 1.

$$a(n,k) = \begin{cases} 0 & k < 0 \text{ or } k \geq n \\ 1 & k = 0 \text{ or } k = n-1 \\ 2\sum_{i=1}^{n-1} a(i,k-1) - b(n-1,k-1) & \text{otherwise} \end{cases}$$

$$b(n,k) = \begin{cases} 0 & k < 1 \text{ or } k \geq n \\ 1 & k = n-1 \\ 2a(n-1,k-1) - b(n-1,k-1) & \text{otherwise} \end{cases}$$

Enumeration via ascents

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Theorem (P and Smith, 2017+)

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} y^{\mathrm{asc}(\pi)} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} a(n,k) x^n y^k = \frac{x(x^2y+1)}{1-x-xy-x^2y-2x^3y^2}$$

$$\begin{aligned}\frac{x(x^2y+1)}{1-x-xy-x^2y-2x^3y^2} &= x + (y+1)x^2 + (y^2+4y+1)x^3 \\ &+ (y^3+8y^2+6y+1)x^4 \\ &+ (y^4+12y^3+20y^2+8y+1)x^5 \\ &+ (y^5+16y^4+48y^3+36y^2+10y+1)x^6 \\ &+ \cdots\end{aligned}$$

Enumeration via ascents

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Theorem (P and Smith, 2017+)

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} y^{\mathrm{asc}(\pi)} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} a(n,k) x^n y^k = \frac{x(x^2y+1)}{1 - x - xy - x^2y - 2x^3y^2}$$

Corollary

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2+1)}{1 - 2x - x^2 - 2x^3}$$

OEIS A224232: $1, 2, 6, 16, 42, 112, 298, 792, \ldots$

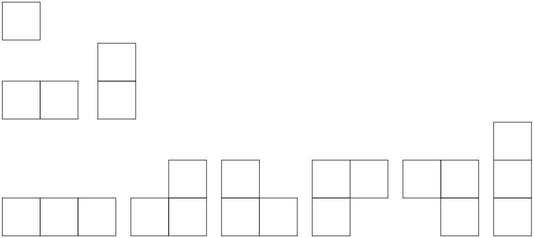
So $|\mathcal{P}_{2,n}| = 2\,|\mathcal{P}_{2,n-1}| + |\mathcal{P}_{2,n-2}| + 2\,|\mathcal{P}_{2,n-3}|$.

Polyominoes

$n = 1$:

$n = 2$:

$n = 3$:



OEIS A001168: $1, 2, 6, 19, 63, 216, 760, \ldots$

(General formula is open)

Polyominoes on a helix

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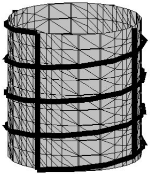
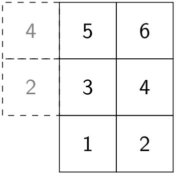
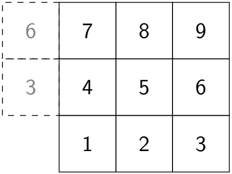


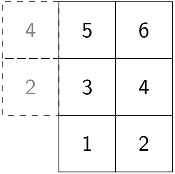


width 2 helix

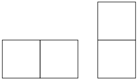


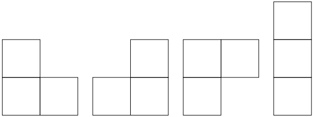
width 3 helix

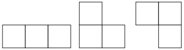
Polyominoes on a helix of width 2



$n = 1$: 

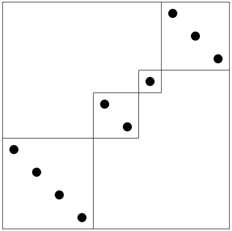
$n = 2$: 

$n = 3$: 

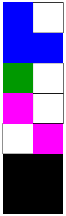
( are all the same polyomino now!)

Sequence: $1, 2, 4, \ldots, 2^{n-1}, \ldots$

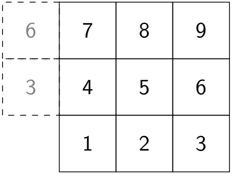
Polyominoes on a helix of width 2

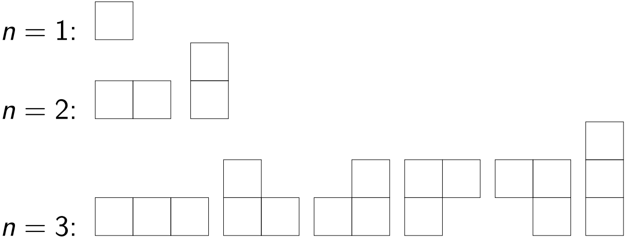


$4 + 2 + 1 + 3$



Polyominoes on a helix of width 3

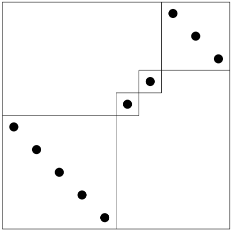




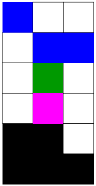
Sequence: $1, 2, 6, 16, 42, 112, \ldots$

(OEIS A224232, same as 2-pop-stack sortable permutations)

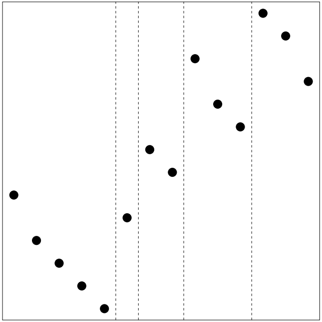
Polyominoes on a helix of width 3



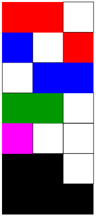
$5 \oplus 1 \oplus 1 \oplus 3$



Polyominoes on a helix of width 3



$5 \boxplus 1 \oplus 2 \oplus 3 \boxplus 3$



Summary

- π is 2-pop-stack sortable iff π avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.
- $$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2+1)}{1-2x-x^2-2x^3}$$
- Bijections!
  - 1-pop-stack sortable permutations are in bijection with polyominoes on a helix of width 2,
  - 2-pop-stack sortable permutations are in bijection with polyominoes on a helix of width 3,

Summary

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- π is 2-pop-stack sortable iff π avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.
- $$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2+1)}{1-2x-x^2-2x^3}$$
- Bijections!
  - 1-pop-stack sortable permutations are in bijection with polyominoes on a helix of width 2,
  - 2-pop-stack sortable permutations are in bijection with polyominoes on a helix of width 3,
  - ...but 3-pop-stack sortable permutations aren't in bijection with polyominoes on a helix of width 4.

References

Valparaiso University

- G. Aleksandrowich, A. Asinowski, and G. Barequet, Permutations with forbidden patterns and polyominoes on a twisted cylinder of width 3. *Discrete Math.* **313** (2013), 1078–1086.
- D. Avis and M. Newborn, On pop-stacks in series. *Utilitas Math.* **19** (1981), 129–140.

Thanks for listening!

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