Quarter Turn Baxter Permutations

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Outline

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What is a Baxter Permutation?

Definition

A *Baxter permutation* is a permutation that, when written in one-line notation, avoids the generalized patterns 3-14-2 and 2-41-3. This is to say that there are no instances of the patterns 3142 or 2413 where the letters representing 1 and 4 are adjacent in the original word.

Example

41352 is a Baxter permutation.

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Number of Baxter Permutations

Theorem (Chung, Graham, Hoggatt, Kleiman)

The number of Baxter permutations of length n is

$$B(n) := \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}$$

For $n = 1, 2, 3 \ldots$, $B(n) = 1, 2, 6, 22, 92, 422, 2074, 10754 \ldots$.

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Theorem (Mallows)

The number of Baxter permutations with k ascents is given by the k^{th} summand, $\frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}$

Multiplication by the longest element (w_0) on either side takes a Baxter permutation of length n with k ascents to a Baxter permutation of length n with $n-k+1$ ascents.

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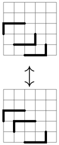
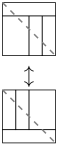
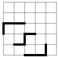
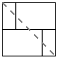
Chart of Examples

Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
2341 $\updownarrow$ 4123	2341 $\updownarrow$ 4123		1 4 6 9 2 5 8 11 3 7 10 12 $\updownarrow$ 1 3 6 10 2 5 8 11 4 7 9 12		2 2 $\updownarrow$ 1 1
3412 $\circlearrowleft$	3142 $\circlearrowleft$	 $\circlearrowleft$	1 3 7 9 2 5 8 11 4 6 10 12 $\circlearrowleft$	 $\circlearrowleft$	2 1 $\circlearrowleft$

Symmetries

- All "Baxter Objects" have a equivariant rotation symmetry.
- Baxter permutations are closed under taking inverses.
- Bousquet-Mèlou gave enumeration for involutive, Fusy bijective proof.
- What about quarter turn rotation of Baxter permutation matrix?

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Outline

Big idea:

- For each $n \in \mathbb{N}$, you have a set of things, $T(n)$.
- Natural restriction map from $\mathrm{Res} : T(n+1) \mapsto T(n)$.
- Define a tree where the parent of $x \in T(n)$ is $\mathrm{Res}(x) \in T(n-1)$.
- Figure out how this tree grows.

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Permutations

- Take $T(n) = S_n$.
- $\mathrm{Res} : S_{n+1} \mapsto S_n$ given by deleting $n+1$.
- There are $n+1$ places we can insert $n+1$ into a permutation of length n, inductively gives $|S_n| = n!$.
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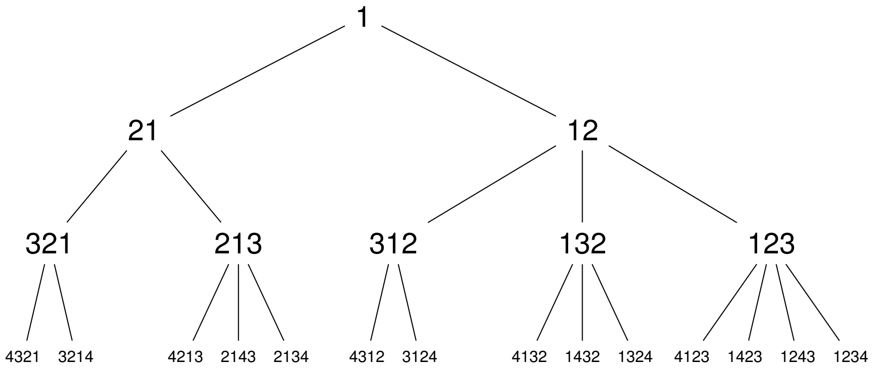
231 Avoiding Permutations

Let $T(n) = Av(231)$



231 Avoiding

Can insert new largest label to left of a left-to-right maximum, or at the end of a word.

41325876

413258769 413259876 413295876 941325876

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4132**58**76

4132**58**76**9** **4**132**59**876 **4**132**9**5876 **9**41325876

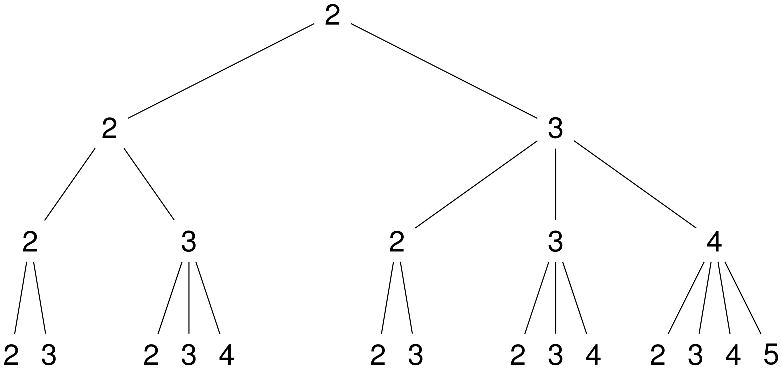

Figure: The beginning of the Catalan tree

Chart of Examples

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1243 ↕ 2134	1243 ↕ 2134	 ↕ 	1 3 6 9 2 5 7 11 4 8 10 12 ↕ 1 3 5 9 2 6 8 11 4 7 10 12	 ↕ 	3 3 ↕ 0 0
1342 ↕ 3124	1342 ↕ 3124	 ↕ 	1 3 6 9 2 4 8 11 5 7 10 12 ↕ 1 3 6 8 2 5 9 11 4 7 10 12	 ↕ 	3 2 ↕ 1 0
↕ 1423 ↕ 2314	↕ 1423 ↕ 2314	↕  ↕ 	↕ 1 3 5 7 2 6 9 11 4 8 10 12 ↕ 1 3 5 9 2 4 7 11 6 8 10 12	↕  ↕ 	↕ 3 1 ↕ 2 0
2341 ↕ 4123	2341 ↕ 4123	 ↕ 	1 4 6 9 2 5 8 11 3 7 10 12 ↕ 1 3 6 10 2 5 8 11 4 7 9 12	 ↕ 	2 2 ↕ 1 1
1324 ↺	1324 ↺	 ↺	1 3 5 7 2 4 9 11 6 8 10 12 ↺	 ↺	3 0 ↺
3412 ↺	3142 ↺	 ↺	1 3 7 9 2 5 8 11 4 6 10 12 ↺	 ↺	2 1 ↺

Baxter Permutations

- Av(231) would be similar with right to left maxima
- Baxter permutations allow both insertion immediately to the left of a left-to-right maxima, and immediately to the right of a right-to-left maxima.

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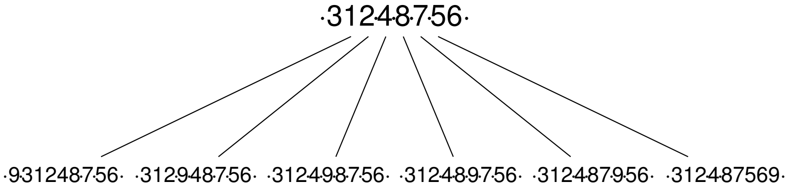
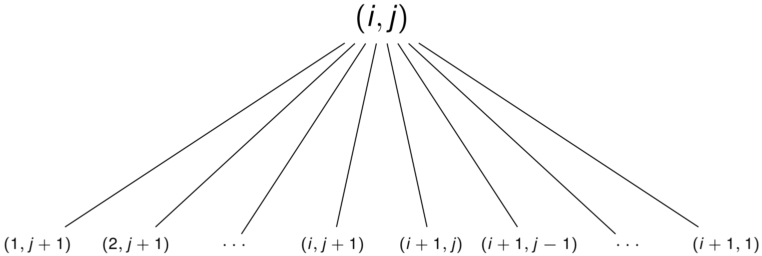


Figure: Branching of generating tree at $w = 31248756$, with insertion points marked.

Baxter Permutations

Rule for generating tree isomorphic to Baxter permutations



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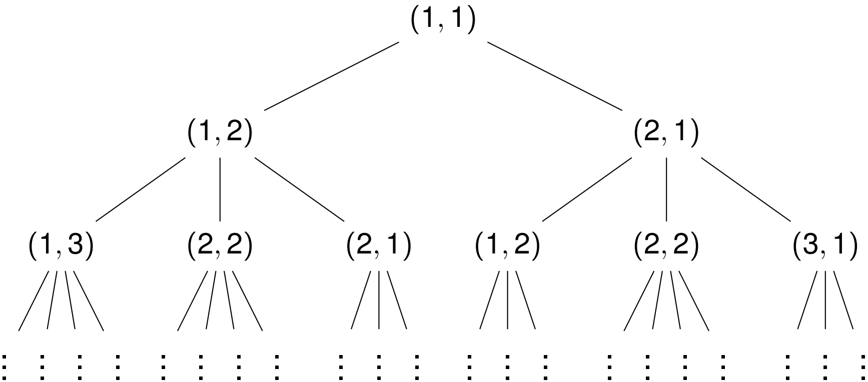

Half Turn Baxter

- Entries in permutation fixed under half turn come in pairs
- If $w_i = j$, then $w_{n+1-i} = n+1-j$
- w=47136825
- Restriction map needs to remove n and 1 (and lower all labels by 1).

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- w=**4**713682**5**
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- w=4**7**1368**2**5
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- If $w_i = j$, then $w_{n+1-i} = n + 1 - j$
- w=47**1**36**8**25
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- If $w_i = j$, then $w_{n+1-i} = n + 1 - j$
- w=471**36**825
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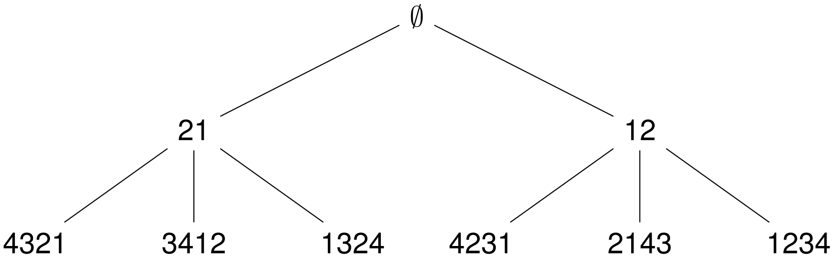

Figure: Generating Tree for Baxter permutations of even length fixed under conjugation by the longest element

Rule for generating tree isomorphic to
Baxter permutations fixed under conjugation by longest element.

$$(i, j)$$

$$(1, j+2) \quad (2, j+1) \quad \cdots \quad (i, j+1) \quad (i+1, j) \quad (i+1, j-1) \quad \cdots \quad (i+2, 1)$$

q=-1 Phenomenon

If we let $n = k + \ell + 1$, $\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$,
$[m]!_q = [m]_q[m-1]_q \ldots [1]_q$, and $[j]_q = 1 + q + \ldots + q^{j-1}$, we have

Theorem (D.)

The number of Baxter objects with parameter k fixed under their natural involution is given by $\frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q} |_{[q=-1]}$

Quarter Turn Baxter

- Equivalent to say if $w_i = j$, then $w_j = n + 1 - i$, $w_{n+1-i} = n + 1 - j$, and $w_{n+1-j} = i$.
- In general it makes a 4-cycle $(i, j, n + 1 - i, n + 1 - j)$.
- Only degenerate case is a single central fixed point.
- Counting descents forces n to be odd, so $n = 4k + 1$.
- Restriction from removing n, 1, w_1, and w_n.

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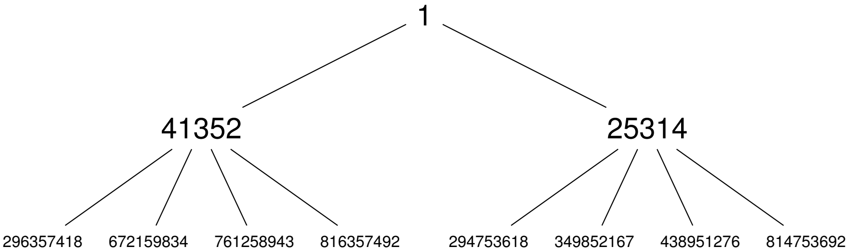

Figure: Start of generating tree for Baxter permutations fixed under 90° rotation.

Quarter Turn Baxter

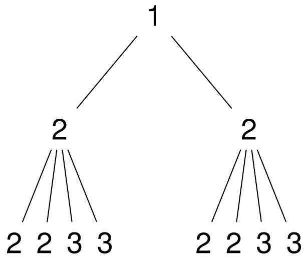

Figure: The beginning of the doubled Catalan tree

Quarter Turn Baxter

Theorem (D.)

The number of Baxter permutations of length n fixed under a quarter turn is equal to $2^k C_k$ *(where* C_k *is the* k^{th} *Catalan number) if* $n = 4k + 1$*, and 0 otherwise*

Combinatorial Interpretation?

Other interesting things with this enumeration (A151374, A052701)

- Obvious: Bicolored Dyck paths
- Less obvious: Rooted Eulerian n-edge map in the plane (Liskovets/Walsh '05)
- More less obvious: Lattice walks in first quadrant with steps $(1, 1)$, $(-1, -1)$, and $(-1, 0)$ starting at origin and ending on y-axis. (Bousquet-Mélou/Mishna '09)

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