On the growth rate of the centrosymmetric permutations in a class

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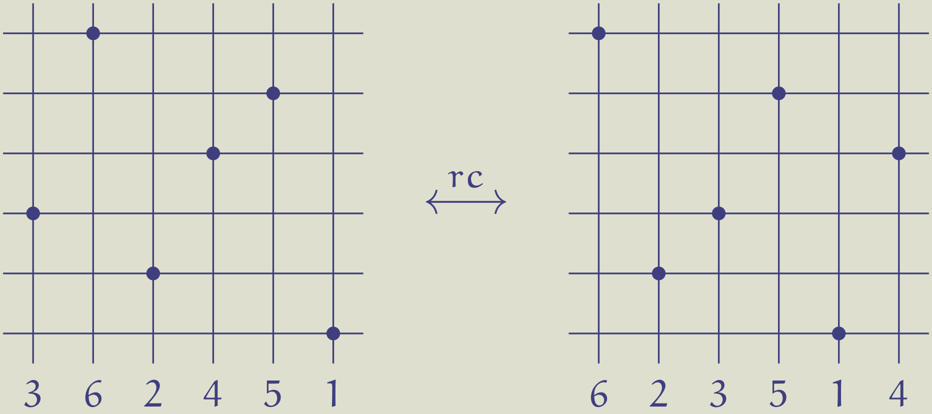
The reverse–complement transformation



- The *reverse–complement* of a permutation π, denoted $\mathrm{rc}(\pi)$, is the permutation obtained by a half-turn rotation of the diagram of π.
- A class $\mathcal{C}$ is rc-*invariant* if $\mathrm{rc}(\mathcal{C}) = \mathcal{C}$.

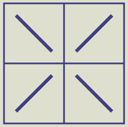
Centrosymmetric permutations

- A permutation π is *centrosymmetric* if $\mathrm{rc}(\pi) = \pi$.
- The number of centrosymmetric permutations of size $2n$ is $(2n)(2n-2)\cdots(4)(2) = 2^n n!$.
- **Definition:** Let $\mathcal{C}^{\mathrm{rc}}$ denote the set of centrosymmetric permutations in $\mathcal{C}$.
- **Definition:** The rc–*growth rate* of $\mathcal{C}$, denoted $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C})$, is the growth rate of $\mathcal{C}^{\mathrm{rc}}_{2n}$; that is,

$$\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}^{\mathrm{rc}}_{2n}|^{1/n}.$$

- At *Permutation Patterns* 2016, Alex Woo presented the following open problem: Which rc-invariant permutation classes $\mathcal{C}$ satisfy $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$?

Example

$\mathcal{C} = \mathrm{Av}(3412, 2143) = \mathrm{Grid}$ 

- $|\mathcal{C}_n| = \binom{2n}{n} - \sum_{m=0}^{n-1} 2^{n-m-1}\binom{2m}{m}$ (Atkinson 1998)
- $\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}_n|^{1/n} = 4.$
- $|\mathcal{C}^{rc}_{2n}| = \binom{2n}{n}$ (T 2016).
- $\mathrm{gr}^{rc}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}^{rc}_{2n}|^{1/n} = 4.$

The big table

R	sum-closed?	gr(Av(R))		$\text{gr}^{rc}(\text{Av}(R))$	
$k \cdots 1$	Yes	$(k-1)^2$		$(k-1)^2$	(Egge 2010)
$231, 312$	Yes	2	(S & S 1985)	2	(Egge 2007)
$321, 3412$	Yes	$\frac{3+\sqrt{5}}{2}$	(West 1996)	$\frac{3+\sqrt{5}}{2}$	(Egge 2007)
$321, 3142$	Yes	$\frac{3+\sqrt{5}}{2}$	(West 1996)	$\frac{3+\sqrt{5}}{2}$	(L & O 2010)
$321, 231, 312$	Yes	$\frac{1+\sqrt{5}}{2}$	(S & S 1985)	$\frac{1+\sqrt{5}}{2}$	(T 2016)
$2413, 3142$	Yes	$3+2\sqrt{2}$	(West 1995)	$3+2\sqrt{2}$	(T 2016)
$4321, 3412$	Yes	4	(K & S 2003)	4	(T 2016)
$4321, 3142$	Yes	$2+\sqrt{3}$	(Vatter 2012)	$2+\sqrt{3}$	(T 2016)
$321, 2143$	No	2	(West 1996)	2	(L & O 2010)
$3412, 2143$	No	4	(Atkinson 1998)	4	(T 2016)
$4231, 1324$	No	$2+\sqrt{2}$	(A, A, & V 2009)	2	(T 2016)
$4321, 2143$	No	$\frac{3+\sqrt{5}}{2}$	(A, A & B 2012)	2	(T 2016)

S & S = Simion & Schmidt;
K & S = Kremer & Shiu;
A, A, & V = Albert, Atkinson, & Vatter;
L & O = Lonoff & Ostroff;
A, A, & B = Albert, Atkinson, & Brignall.

I. **Counterexamples**

II. **Geometric grid classes and generalized grid classes**

III. **$\oplus$-closed classes**

Counterexamples

- Suppose a class $\mathcal{D}$ is *not* rc-invariant. Then

$$\mathcal{D} \cap \mathrm{rc}(\mathcal{D}) \subsetneqq \mathcal{D} \quad \text{but} \quad (\mathcal{D} \cap \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}} = \mathcal{D}^{\mathrm{rc}}.$$

- Example:
  - $\mathcal{D} = \mathrm{Av}(312)$ (growth rate 4).
  - $\mathcal{D} \cap \mathrm{rc}(\mathcal{D}) = \mathrm{Av}(312, 231)$ (growth rate 2).
  - $\mathcal{D}^{\mathrm{rc}} = (\mathcal{D} \cap \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}} = \mathrm{Av}(312, 231)^{\mathrm{rc}}$, and $\mathrm{gr}^{\mathrm{rc}}(\mathcal{D}) = 2$.

Counterexamples from unions

- Suppose a class $\mathcal{D}$ is *not* rc-invariant. Then

$$\mathcal{D} \cap \mathrm{rc}(\mathcal{D}) \subsetneqq \mathcal{D} \quad \text{but} \quad (\mathcal{D} \cap \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}} = \mathcal{D}^{\mathrm{rc}}.$$

- Example:
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  - $\mathcal{D}^{\mathrm{rc}} = (\mathcal{D} \cap \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}} = \mathrm{Av}(312, 231)^{\mathrm{rc}}$, and $\mathrm{gr}^{\mathrm{rc}}(\mathcal{D}) = 2$.
- Furthermore, $\mathcal{D} \cup \mathrm{rc}(\mathcal{D})$ is rc-invariant, and

$$\mathcal{D} \cap \mathrm{rc}(\mathcal{D}) \subsetneqq \mathcal{D} \cup \mathrm{rc}(\mathcal{D}) \quad \text{but} \quad (\mathcal{D} \cap \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}} = (\mathcal{D} \cup \mathrm{rc}(\mathcal{D}))^{\mathrm{rc}}.$$

  So, setting $\mathcal{C} = \mathcal{D} \cup \mathrm{rc}(\mathcal{D})$, we often find that $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) < \mathrm{gr}(\mathcal{C})$.

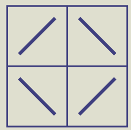
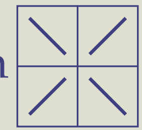
Counterexamples from unions

$$\mathcal{C} = \mathcal{D} \cup \mathrm{rc}(\mathcal{D}).$$

$\mathcal{D}$	$\mathrm{gr}(\mathcal{C})$	$\mathrm{gr}^{\mathrm{rc}}(\mathcal{C})$	basis of $\mathcal{C}$
$\mathrm{Av}(312)$	4 (Knuth 1973)	2 (Egge 2007)	$\left\{\begin{matrix} 2413, 3142, 3412, \\ 4231, 231645, 312564 \end{matrix}\right\}$
$\mathrm{Av}(4123)$	9 (Stankova 1996)	4 (T 2017)	29 patterns
$\mathrm{Av}(4312)$	9 (West 1990)	$2+\sqrt{5}$ (T 2017)	69 patterns

Counterexamples not from unions

- Let $\mathcal{C}$ be either one of these two *geometric grid classes*:

$\mathcal{C} = \mathrm{Geom}$  or $\mathcal{C} = \mathrm{Geom}$ 

- Every geometric grid class is *atomic* (AABRV 2013), and every rc-invariant geometric grid class is generated by its centrosymmetric elements.
- $\mathrm{gr}(\mathcal{C}) = 2 + \sqrt{2}$, but $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) = 2$.

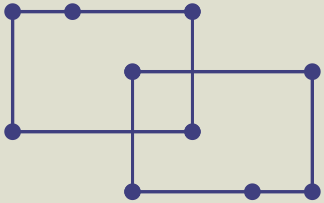
II. Geometric grid classes and generalized grid classes

Geometric grid classes

Let M be a centrosymmetric $\{0, 1, -1\}$-matrix.

- The *cell graph* of M is the graph whose vertices are the non-zero entries of M, where two entries are adjacent if (1) they share a row or column and (2) there are no non-zero entries between them in their row or column.
- **Example:**

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \mapsto$$



Geometric grid classes and generalized grid classes

Let M be a centrosymmetric $\{0, 1, -1\}$-matrix with cell graph G.

- **Theorem on geometric grid classes** (T 2017): Each of the following statements implies the next one:
  - *(i)* G is a forest;
  - *(ii)* For every vertex v of G, there is no path from v to $\mathrm{rc}(v)$;
  - *(iii)* $\mathrm{gr}^{\mathrm{rc}}(\mathrm{Geom}(M)) = \mathrm{gr}(\mathrm{Geom}(M))$.
- An analogous theorem holds for generalized grid classes.
- **Conjecture**: If $\mathcal{C}$ is the monotone grid class of M, then $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$.

III. $\oplus$-closed classes

$\oplus$-closed classes

Assume $\mathcal{C}$ is rc-invariant and $\oplus$-closed.

- **Definition:** Let $\widetilde{\mathcal{C}}$ denote the set of $\oplus$-indecomposable permutations in $\mathcal{C}$, and let $\widetilde{\mathcal{C}}^{rc}$ denote the set of centrosymmetric $\oplus$-indecomposable permutations in $\mathcal{C}$.
- **Definition:** Let $\mathrm{gr}(\widetilde{\mathcal{C}})$ denote the growth rate of $\widetilde{\mathcal{C}}_n$, and let $\mathrm{gr}^{rc}(\widetilde{\mathcal{C}})$ denote the growth rate of $\widetilde{\mathcal{C}}^{rc}_{2n}$; that is,

$$\mathrm{gr}(\widetilde{\mathcal{C}}) = \lim_{n\to\infty} \left|\widetilde{\mathcal{C}}_n\right|^{1/n} \qquad \text{and} \qquad \mathrm{gr}^{rc}(\widetilde{\mathcal{C}}) = \lim_{n\to\infty} \left|\widetilde{\mathcal{C}}^{rc}_{2n}\right|^{1/n}.$$

- **Theorem** (T 2017): $\mathrm{gr}^{rc}(\mathcal{C}) = \max\left\{\mathrm{gr}(\mathcal{C}), \mathrm{gr}^{rc}(\widetilde{\mathcal{C}})\right\}$.
- **Corollary** (T 2017): $\mathrm{gr}^{rc}(\mathcal{C}) \geqslant \mathrm{gr}(\mathcal{C})$.

$\oplus$-closed classes

Assume $\mathcal{C}$ is rc-invariant and $\oplus$-closed. Let $\xi \approx 2.31$ be the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$.

- **Theorem** (T 2017): Each of the following statements implies the next one:

$$\begin{array}{rlrl} \textit{(i)} & \mathrm{gr}(\mathcal{C}) \leqslant \xi; & \textit{(ii)} & |\widetilde{\mathcal{C}}_n| \text{ is bounded;} \\ \textit{(iii)} & \mathrm{gr}(\widetilde{\mathcal{C}}) \text{ is either } 0 \text{ or } 1; & & \\ \textit{(iv)} & \mathrm{gr}^{rc}(\widetilde{\mathcal{C}}) \leqslant \mathrm{gr}(\mathcal{C}); & \textit{(v)} & \mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C}). \end{array}$$

  - (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are easy, and (iv) $\Rightarrow$ (v) follows easily from the previous theorem.

How to prove (i) $\Rightarrow$ (ii)

Pantone & Vatter, "Growth rates of permutation classes":

sequence	growth rate is the greatest real root of	bound
$1,1,2,4,3,3,2,1$	$x^5-2x^4-x^2-x-1$	$\xi \approx 2.30522$
$1,1,2,4,3,3,3$	$x^7-x^6-x^5-2x^4-4x^3-3x^2-3x-3$	> 2.30688
$1,1,2,4,4,1,1,1,1,1,1$	$x^{11}-x^{10}-x^9-2x^8-4x^7-4x^6-x^5-x^4-x^3-x^2-x-1$	> 2.30525
$1,1,2,4,4,2$	$x^6-x^5-x^4-2x^3-4x^2-4x-2$	> 2.30692
$1,1,2,4,5$	$x^5-x^4-x^3-2x^2-4x-5$	> 2.30902
$1,1,2,5,2,1,1$	$x^6-2x^5+x^4-3x^3-2x^2-1$	> 2.30790
$1,1,2,5,2,2$	$x^6-x^5-x^4-2x^3-5x^2-2x-2$	> 2.31179
$1,1,2,5,3$	$x^5-x^4-x^3-2x^2-5x-3$	> 2.31392
$1,1,3,3,1,1,1,1,1,1$	$x^{10}-x^9-x^8-3x^7-3x^6-x^5-x^4-x^3-x^2-x-1$	> 2.30528
$1,1,3,3,2$	$x^5-x^4-x^3-3x^2-3x-2$	> 2.30939
$1,1,3,4$	x^3-2x^2+x-4	> 2.31459

Table 1: Short legal sequences leading to growth rates of at least ξ.

sequence	growth rate is the greatest real root of
$1,1,2,3,4^\infty$	$x^5-2x^4-x^2-x-1$
$1,1,2,3,4^i,5,3,3,2,1$	$x^5-2x^4-x^2-x-1$
$1,1,2,3,4^i,5,3,3,3$	$x^{i+4}\left(x^5-2x^4-x^2-x-1\right)-x^4+2x^3+3$
$1,1,2,3,4^i,5,4,1,1,1,1,1,1$	$x^{i+8}\left(x^5-2x^4-x^2-x-1\right)-x^8+x^7+3x^6+1$
$1,1,2,3,4^i,5,4,2$	$x^{i+3}\left(x^5-2x^4-x^2-x-1\right)-x^3+x^2+2x+2$
$1,1,2,3,4^i,5,5$	$x^{i+2}\left(x^5-2x^4-x^2-x-1\right)-x^2+5$
$1,1,2,3,4^i,6,2,1,1$	$x^{i+4}\left(x^5-2x^4-x^2-x-1\right)-2x^4+4x^3+x^2+1$
$1,1,2,3,4^i,6,2,2$	$x^{i+3}\left(x^5-2x^4-x^2-x-1\right)-2x^3+4x^2+2$
$1,1,2,3,4^i,6,3$	$x^{i+2}\left(x^5-2x^4-x^2-x-1\right)-2x^2+3x+3$
$1,1,2,3,4^i,7,1$	$x^{i+2}\left(x^5-2x^4-x^2-x-1\right)-3x^2+6x+1$
$1,1,2,3,4^i,8$	$x^{i+1}\left(x^5-2x^4-x^2-x-1\right)-4x+8$

Table 2: Long legal sequences leading to growth rates of at least ξ.

Conjectures

- **Conjecture:** If $\mathcal{C}$ is rc-invariant and $\oplus$-closed, then $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$.
- **Conjecture:** If $\mathcal{C}$ is rc-invariant, then $\mathrm{gr}^{\mathrm{rc}}(\mathcal{C}) \leqslant \mathrm{gr}(\mathcal{C})$.
- The second conjecture implies the first.

THANK YOU!