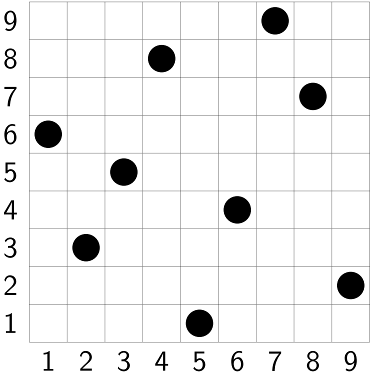
Juxtaposing Catalan classes with monotone ones

Jakub Sliačan (joint work with Robert Brignall)

Permutation Patterns 2017

View permutations as drawings

635814972 $\longleftrightarrow$



Enumerating permutation classes

Class

Collection of permutations closed under containment (if $\pi \in \mathcal{C}$, then all subpermutations $\sigma \subset \pi$ are also in $\mathcal{C}$)

Enumeration

Determining the number of permutations of each length in $\mathcal{C}$

Goal: enumerate simple juxtaposition classes

Catalan class

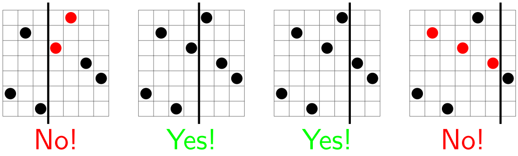
A class of permutations that avoid one of the length 3 patterns: 123,132,213,231,312,321.

$\mathrm{Av}(abc|xy) = \boxed{\text{Cat} \mid \mathcal{M}}$

Let $\mathcal{C}_1, \mathcal{C}_2$ be permutation classes. Their *juxtaposition* $\mathcal{C} = \mathcal{C}_1|\mathcal{C}_2$ is the class of all permutations that can be partitioned such that the left part is a pattern from $\mathcal{C}_1$ and the right part is the pattern from $\mathcal{C}_2$.

Interested in: $\mathcal{C}_1$ = Catalan class, $\mathcal{C}_2$ = Monotone class.

Example: $2615743 \in \mathrm{Av}(321|12)$, witnessed by the middle two partitions.



Today

$$\begin{array}{ccc}
\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} & \xleftrightarrow{\theta} & \underline{\mathrm{Av}(321|12)},\ \mathrm{Av}(123|21) \\
\mathrm{Av}(123|12),\ \underline{\mathbf{Av(321|21)}} & \xleftrightarrow{\psi} & \underline{\mathrm{Av}(231|21)},\ \mathrm{Av}(213|12) \\
\mathrm{Av}(132|12),\ \underline{\mathbf{Av(312|21)}} & \xleftrightarrow{\phi} & \underline{\mathrm{Av}(312|12)},\ \mathrm{Av}(132|21)
\end{array}$$

Enumerated by Bevan and Miner, respectively

Enumerated (here)

Bijections θ, ψ, ϕ between underlined classes (given here)

Why these juxtapositions?

Because they show up, e.g.

- Bevan enumerated $\mathrm{Av}(231|12)$ (or its symmetry) as a step to enumerating $\mathrm{Av}(4213, 2143)$.
- Miner enumerated $\mathrm{Av}(123|21)$ (or its symmetry) as a step to enumerating $\mathrm{Av}(4123, 1243)$.

Because they are "simplest" grid classes

- Murphy, Vatter (2003)
- Albert, Atkinson, and Brignall (2011)
- Vatter, Watton (2011)
- Brignall (2012)
- Albert, Atkinson, Bouvel, Ruškuc, and Vatter (2013)
- Bevan (2016)

We can't enumerate this

$$
\begin{array}{|c|c|c|c|c ccc|}
\hline
\mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & & & & & \mathcal{C}_{1m} \\
\hline
\mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & & & & & \mathcal{C}_{2m} \\
\hline
\mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & & & & \cdots & \mathcal{C}_{3m} \\
\hline
 & & & & & & & \\
\hline
 & & \vdots & & & & \ddots & \\
\mathcal{C}_{n1} & \mathcal{C}_{n2} & \mathcal{C}_{n3} & & & & & \mathcal{C}_{nm} \\
\hline
\end{array}
$$

Even if $\mathcal{C}_{ij}$ are permutation classes that we CAN enumerate

. . . or this

$$
\begin{array}{cccccc}
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & \mathcal{M} \\
\mathcal{M} & \mathcal{M} & \color{red}{\mathcal{C}} & \mathcal{M} & & \mathcal{M} \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \\
 & & & & & \\
 & & \vdots & & \ddots & \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M}
\end{array}
$$

$\mathcal{M}$ monotone classes, $\mathcal{C}$ non-monotone class

. . . actually, not even this

$$
\begin{array}{|c|c|c|c|c ccc|}
\hline
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M} \\
\hline
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M} \\
\hline
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \cdots & \mathcal{M} \\
\hline
 & & & & & & & \\
\hline
 & & & & & & & \\
 & & \vdots & & & & \ddots & \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & & & & & \mathcal{M} \\
\hline
\end{array}
$$

$\mathcal{M}$ monotone classes

But! we know their growth rates = (spectral radius)^2 of the row-column graph [Bev15a].

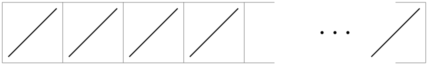
... also ...

these have rational generating functions [AAB⁺13]

$$\mathrm{Geom}\begin{pmatrix} \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & \cdots & \mathcal{M} \\ & & & & & & \\ & & \vdots & & & \ddots & \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M} \end{pmatrix}$$

. . . and . . .

generating functions conjectured for monotone increasing strips [Bev15b]



. . . and . . .

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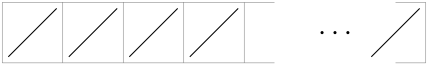

Idea: be less ambitious

So...

Enumerate juxtapositions of monotone and Catalan cells

We'll look at the blue parts

$$\begin{array}{ccc}
\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} & \xleftrightarrow{\theta} & \mathrm{Av}(123|21),\ \underline{\mathrm{Av}(321|12)} \\
\mathrm{Av}(123|12),\ \underline{\mathbf{Av(321|21)}} & \xleftrightarrow{\psi} & \mathrm{Av}(213|12),\ \underline{\mathrm{Av}(231|21)} \\
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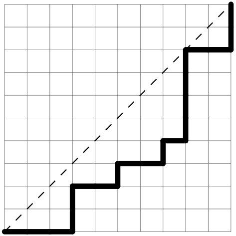
\end{array}$$

Dyck paths

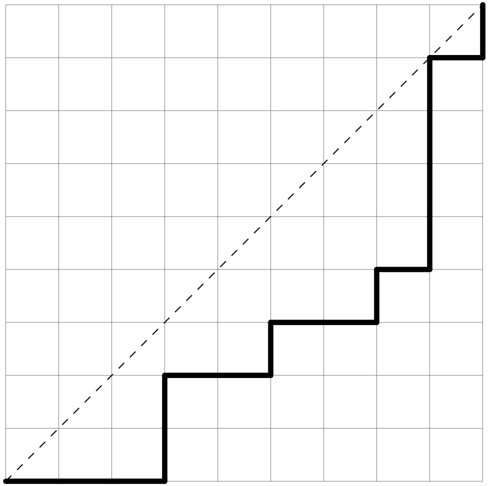
Dyck path

A Dyck path of length $2n$ is a path on the integer grid from top right to bottom left. Each step is either Down (D) or Left (L) and the path stays below the diagonal.

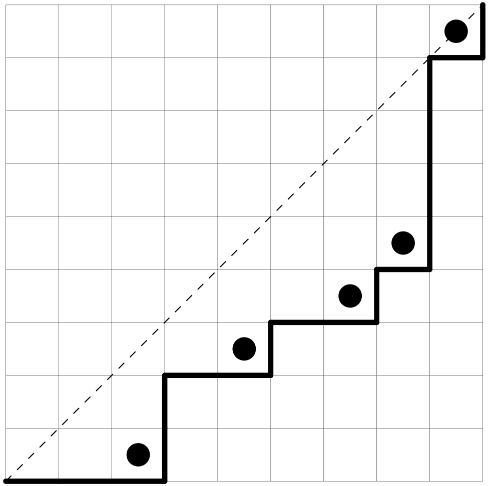
Example



231-avoiders and Dyck paths



231-avoiders and Dyck paths



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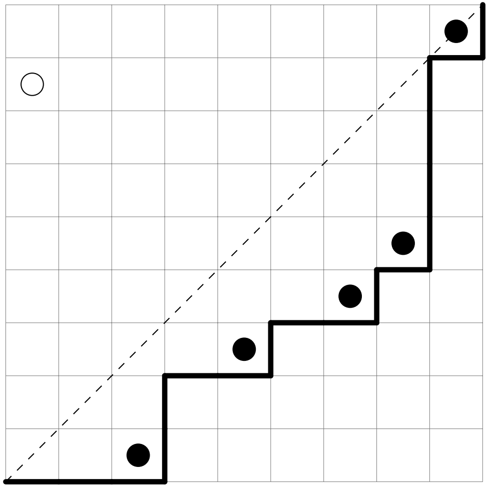

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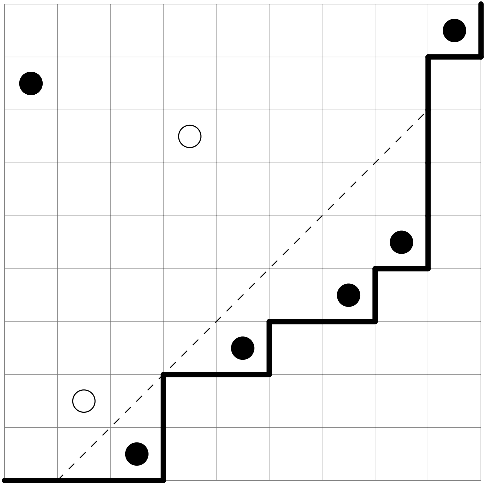

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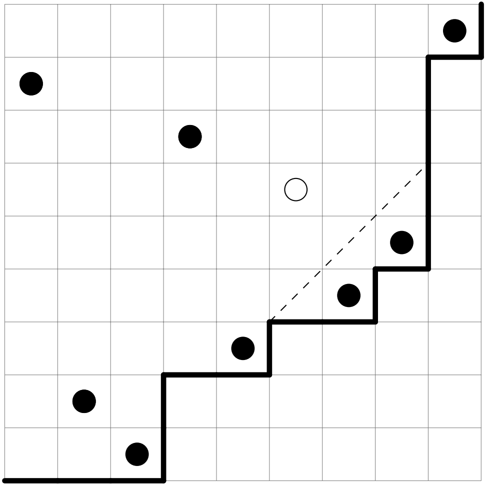

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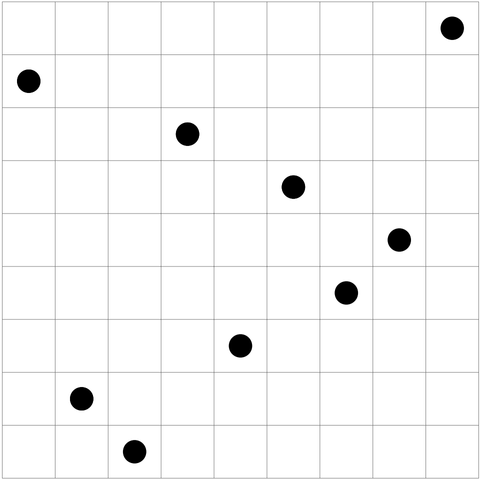

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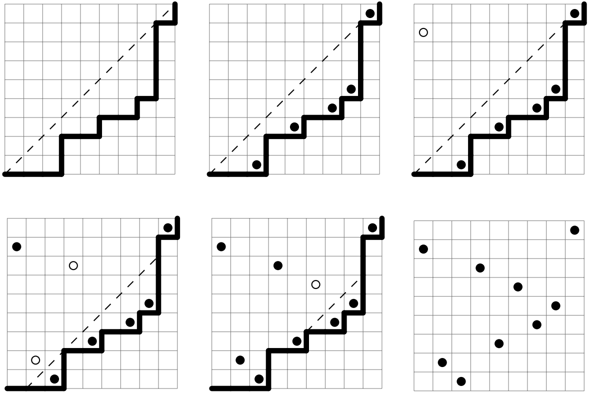
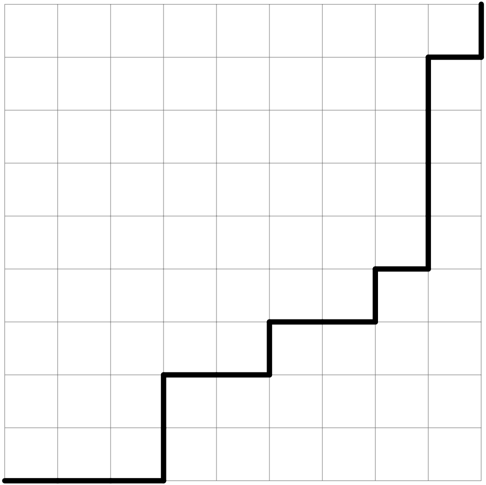
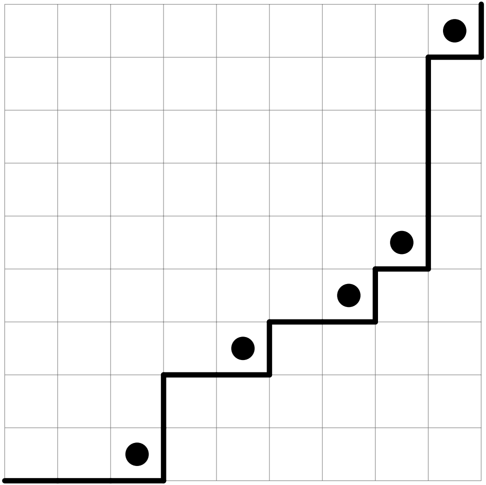


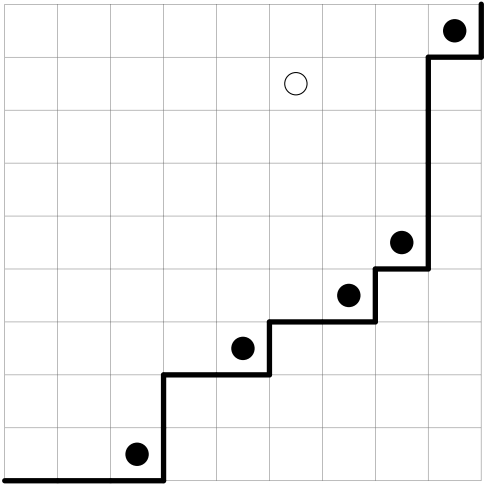
321-avoiders and Dyck paths



321-avoiders and Dyck paths



321-avoiders and Dyck paths



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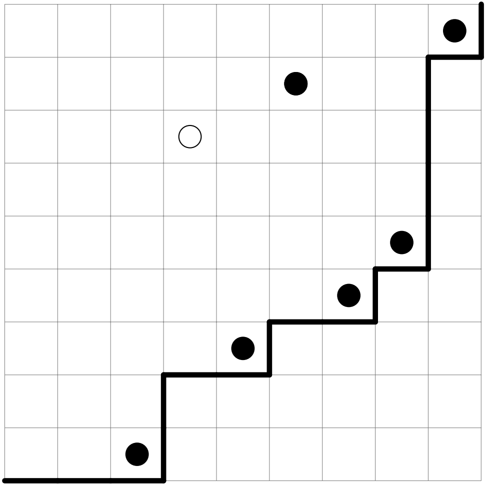

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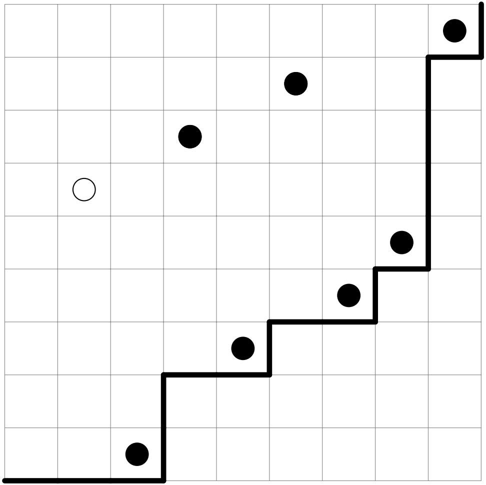

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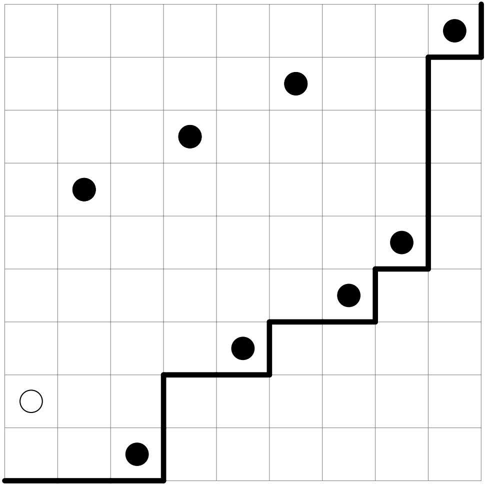

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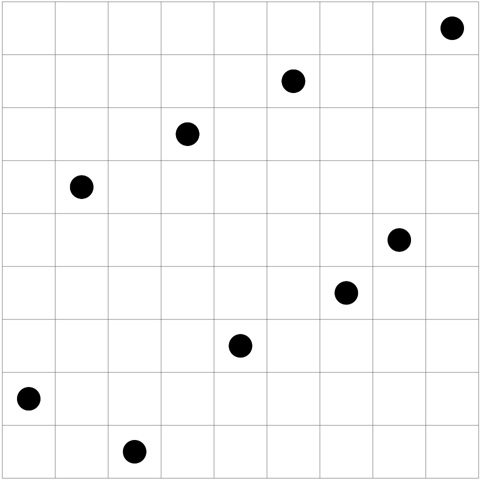

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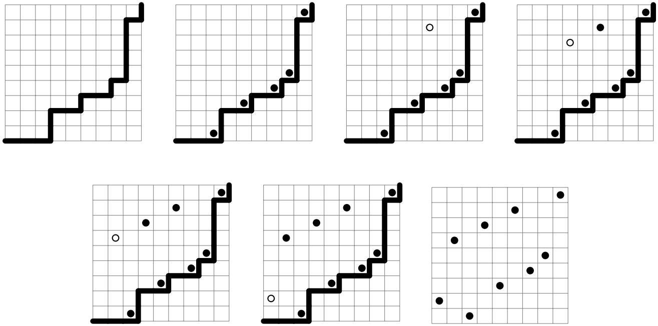


Context-free grammars

Definition

A context-free grammar (CFG) is a formal grammar that describes a language consisting of only those words which can be obtained from a starting string by repeated use of permitted production rules/substitutions.

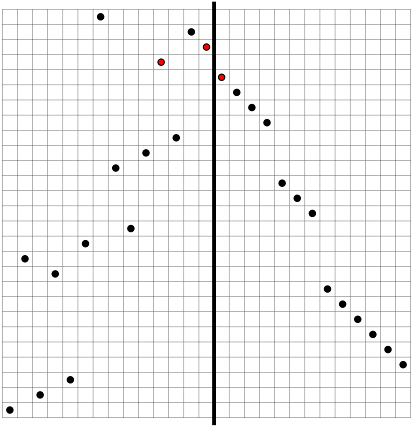
Example: Catalan class by itself (as a CFG)

- **variables:** C
- **characters:** $\epsilon, \mathsf{D}, \mathsf{L}$
- **relations:** $\mathsf{C} \to \epsilon \mid \mathsf{DCLC}$

This gives the following equation:

$$c = 1 + zc^2.$$

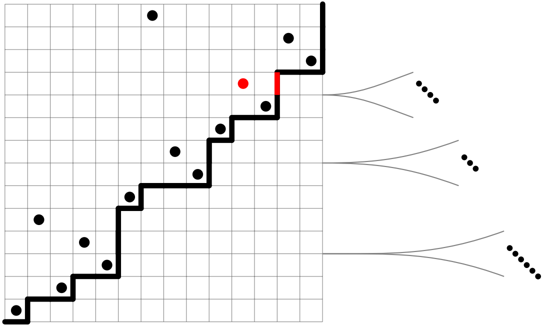
$Av(231|12)$ – gridline greedily right



griddable $\rightarrow$ gridded

$Av(231|12)$ – decorating Dyck paths

- insert point sequences under vertical steps
- first sequence (from top) under first vertical step after a horizontal step occured – first 12 occured



$\mathrm{Av}(231|12)$ – context-free grammar

- L – left step
- D – down step before any left steps occured
- **D** – down step after left step already occured

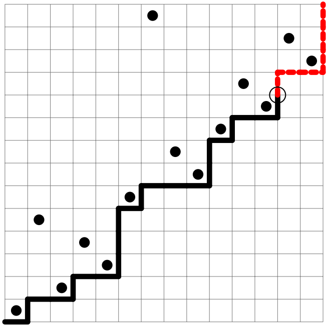
We denote by **C** a Dyck path over letters L and **D**, while C is a standard Dyck path over L and D.

$$\mathsf{S} \to \epsilon \mid \mathsf{DSL}\mathbf{C}$$
$$\mathbf{C} \to \epsilon \mid \mathbf{DC}\mathsf{L}\mathbf{C}$$

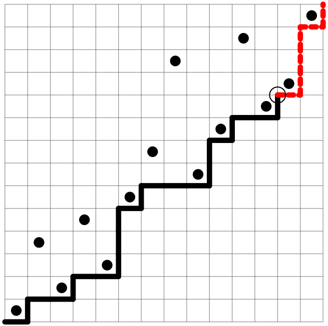
$$s = 1 + zs\mathbf{c}$$
$$\mathbf{c} = 1 + tz\mathbf{c}^2$$

$\mathrm{Av}(321|21)$ and $\mathrm{Av}(312|21)$ "similar".

Articulation point



(a) in $\mathrm{Av}(231)$



(b) in $\mathrm{Av}(321)$

common black part, unique red parts

Bijection $\theta : \mathrm{Av}(231|12) \to \mathrm{Av}(321|12)$

Idea

Choose a good bijection $\theta_0 : \mathrm{Av}(231) \to \mathrm{Av}(321)$. Then extend it to θ by preserving the RHS. $\square$

Bijection $\phi : \mathrm{Av}(312|21) \to Av(312|12)$

Dyck paths $\mathcal{P}$ representing $\mathrm{Av}(312)$.

Recipe

1. Decompose $\mathcal{P}$ into excursions: $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_k$.
2. Identify *middle* part $\mathcal{P}_i$. Where pts on the RHS start.
3. Construct $\mathcal{P}'$ as: $\mathcal{P}_{i+1} \oplus \cdots \oplus \mathcal{P}_n \oplus \mathcal{P}_i \oplus \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_{i-1}$
4. Substitute $\mathcal{P}_i'$ for $\mathcal{P}_i$, where the order of vertical steps in $\mathcal{P}_i'$ is reversed (together with sequences of points on the RHS that go with those vertical steps).

Reversible and resulting Dyck path corresponds to a permutation from $\mathrm{Av}(312|12)$.

Summary

$$\begin{array}{rcl}
\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} & \overset{\theta}{\longleftrightarrow} & \mathrm{Av}(123|21),\ \underline{\mathrm{Av}(321|12)} \\
\mathrm{Av}(123|12),\ \underline{\mathbf{Av(321|21)}} & \overset{\psi}{\longleftrightarrow} & \mathrm{Av}(213|12),\ \underline{\mathrm{Av}(231|21)} \\
\mathrm{Av}(132|12),\ \underline{\mathbf{Av(312|21)}} & \overset{\phi}{\longleftrightarrow} & \mathrm{Av}(132|21),\ \underline{\mathrm{Av}(312|12)}
\end{array}$$

Next

- non-Catalan juxtaposed with monotone
- iterated juxtapositions of monotone
- 2-dim monotone grid classes without cycles

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