Fixed points of pattern-avoiding involutions

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Joint work with
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and
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Outline of talk:

- Analytic combinatorics with bivariate generating functions.
- Standard Young tableaux and involutions avoiding monotone patterns.

Notation:

- $\mathbf{Av}(\rho)$ ($\mathbf{Av}_n(\rho)$) $=$ ρ-avoiding permutations (of size n).
- $\mathbf{Iv}(\rho)$ ($\mathbf{Iv}_n(\rho)$)$=$ ρ-avoiding involutions (of size n).
- $fp(\sigma)$ $=$ number of fixed points of σ.
- For a permutation class $\mathcal{C}$, the bivariate generating function wrt fixed points is given by:

$$F_{\mathcal{C}}(x,t) = \sum_{\sigma \in \mathcal{C}} x^{fp(\sigma)} t^{|\sigma|}.$$

From Elizalde (2004), for $\tau = 321, 132$, or 213,

$$F_{\mathbf{Av}(\tau)}(x,t) = \sum_{\sigma \in \mathbf{Av}(\tau)} x^{fp(\sigma)} t^{|\sigma|} = \frac{2}{1 + 2t(1-x) + \sqrt{1-4t}}.$$

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By Analytic Combinatorics (Flajolet and Sedgewick 2009):

$$[t^n] F_{\mathbf{Av}(321)}(x,t) \sim -[t^n] \frac{8}{(3-x)^2} \sqrt{1-4t} \sim \frac{4^{n+1}}{(3-x)^2 \sqrt{\pi n^3}}$$

so

$$\frac{[t^n] F_{\mathbf{Av}(321)}(x,t)}{[t^n] F_{\mathbf{Av}(321)}(1,t)} \to \frac{4}{(3-x)^2} = \sum_{k=0}^{\infty} \frac{4}{9}(k+1) \left(\frac{1}{3}\right)^k x^k.$$

Enumeration involutions avoiding patterns of length 3 (Simion and Schmidt, 1985)

- For $\tau = 123, 321, 132, 213$, $|\mathbf{Iv}_n(\tau)| = \binom{n}{\lfloor n/2 \rfloor}$.
- For $\tau = 231, 312$, $|\mathbf{Iv}_n(\tau)| = 2^{n-1}$

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Asymptotic enumeration of involutions avoiding monotone patterns (Regev 1981)

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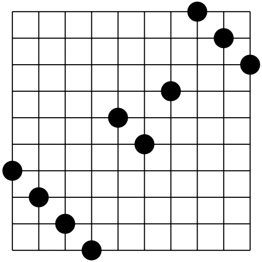
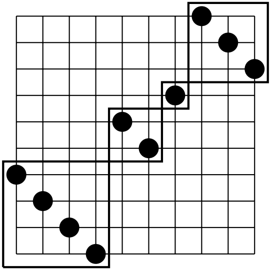
Involutions avoiding patterns of length 4 (Bóna, Homberger, Pantone, and Vatter 2014)

$$\sigma = 4\ 3\ 2\ 1\ 6\ 5\ 7\ 10\ 9\ 8 \in \mathbf{Iv}_n(231)$$



$$\sigma = 4\ 3\ 2\ 1\ 6\ 5\ 7\ 10\ 9\ 8 \in \mathbf{lv}_n(231)$$
$$\alpha = (4, 2, 1, 3).$$



$$F_{\mathbf{lv}(231)} = F_{\mathbf{lv}(312)} = \frac{1 - t^2}{1 - 2t^2 - xt}.$$

Theorem (Miner, Rizzolo, S. 2017)

Let Π_n *denote a uniform random element from* $\mathbf{Iv}_n(231) = \mathbf{Iv}_n(312)$. *Then*

$$\frac{fp(\Pi_n) - \frac{1}{3}n}{\sqrt{8n/27}} \to_d Z,$$

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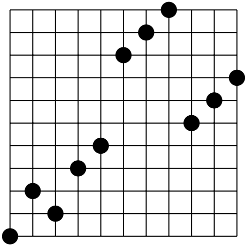
Prove using Theorem IX.9 of Flajolet and Sedgewick on

$$F_{\mathbf{Iv}(231)} = \frac{1-t^2}{1-2t^2-xt}.$$

$$\mathbb{E}(fp(\Pi_n)) = (n/3) + O(1).$$

$$Var(fp(\Pi_n)) = (8n/27) + O(1).$$

$$\sigma = 1\ 3\ 2\ 4\ 5\ 9\ 10\ 11\ 6\ 7\ 8 \in \mathbf{Iv}_{11}(321).$$



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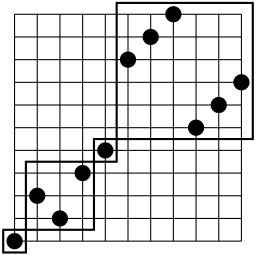

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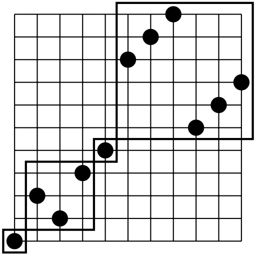

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Theorem (Miner, Rizzolo, S. 2017)

For $\tau = 321, 132,$ or 213:

$$F_{\mathbf{lv}(\tau)}(x,t) = \frac{2}{1 - 2xt + \sqrt{1-4t^2}} = \frac{F_{\mathbf{lv}(\tau)}(0,t)}{1 - xtF_{\mathbf{lv}(\tau)}(0,t)}.$$

If Π_n is a uniformly random element of $\mathbf{lv}_n(\tau)$, then

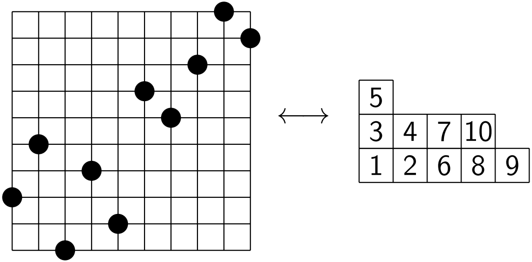
$$\sqrt{\frac{1}{n}}fp(\Pi_n) \to_d X,$$

where X follows a Rayleigh(1) distribution with density function $f(x) = xe^{-x^2/2}$.

Avoiding monotone patterns and standard Young Tableaux

Classic Results

- $\tau_k = (k+1)k \cdots 21$, $\rho_k = 12 \cdots k(k+1)$.



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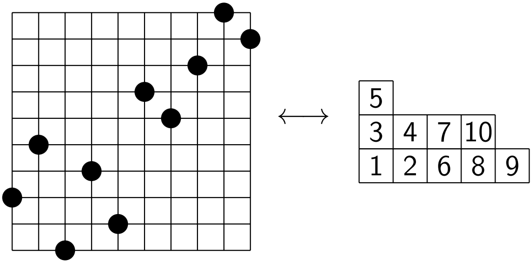
- $\mathbf{Iv}_n(\tau_k)$ (or $\mathbf{Iv}_n(\rho_k)$) is in bijection with standard Young tableaux with at most k rows (or k columns) through RSK. [Schensted, 1961]



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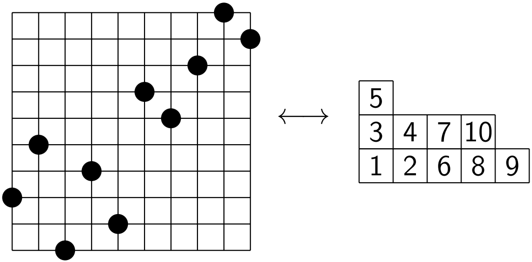
- Fixed points of π = number of odd columns. [Schutzenberger, 1963]



Theorem (Matsumoto 2008)

Let $(\Lambda_i)_{1 \le i \le k}$ be the ranked eigenvalues of a traceless $k \times k$ GOE matrix. For an involution π, let $\lambda_i(\pi)$ denote the length of the ith row of the tableaux of π under RSK. For Π_n chosen uniformly from $\mathbf{Iv}_n(\tau_k)$,

$$\left(\sqrt{\frac{k}{n}}\left(\lambda_i(\Pi_n) - \frac{n}{k}\right)\right)_{1 \le i \le k} \longrightarrow_d (\Lambda_i)_{1 \le i \le k}.$$

Moreover, for any fixed $d > 0$,

$$\mathbb{P}\left(\min_{2 \le i \le k}\{\lambda_{i-1}(\Pi_n) - \lambda_i(\Pi_n)\} < d\right) \to 0.$$

Theorem (Miner, Rizzolo, S. 2017)

Fix $k \in \{2, 3, \dots\}$. Let $(\Lambda_i)_{1 \le i \le k}$ be the ranked eigenvalues of a traceless $k \times k$ GOE matrix and let Π_n be a uniformly random element of $\mathbf{lv}_n((k+1)k \cdots 321)$.

1. *If k is even then*

$$\sqrt{\frac{k}{n}} fp(\Pi_n) \to_d \sum_{j=1}^{k} (-1)^{j+1} \Lambda_j.$$

2. *If k is odd then*

$$\sqrt{\frac{k}{n}} \left(fp(\Pi_n) - \frac{n}{k} \right) \to_d \sum_{j=1}^{k} (-1)^{j+1} \Lambda_j.$$

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$$\sqrt{\frac{k}{n}} \left(fp(\Pi_n) - \frac{n}{k} \right) \to_d \sum_{j=1}^{k} (-1)^{j+1} \Lambda_j.$$

Note $fp(\Pi_n) = \sum_{j=1} \lambda_{2j-1}(\Pi_n) - \lambda_{2j}(\Pi_n)$.

Theorem (Miner, Rizzolo, S. 2017)

If Π_n *is a uniformly random element in* $\mathbf{Iv}_n(123\cdots k(k+1))$ *then*

$$fp(\Pi_{2n}) \to_d X_{even}$$

and

$$fp(\Pi_{2n-1}) \to_d X_{odd}\,,$$

where X_{even} *has density function given by*

$$\mathbb{P}(X_{even} = i) = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is even}\,, \\ 0 & i \text{ is odd}\,, \end{cases}$$

and X_{odd} *has density function given by*

$$\mathbb{P}(X_{odd} = i) = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is odd}\,, \\ 0 & i \text{ is even}\,. \end{cases}$$

Markov chain C with state space $S = \{0, 1, \ldots, k\}$, and transition matrix P with probabilities

$$P_{i,j} = \begin{cases} \frac{i}{k} & j = i - 1 \\ 1 - \frac{i}{k} & j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Lemma

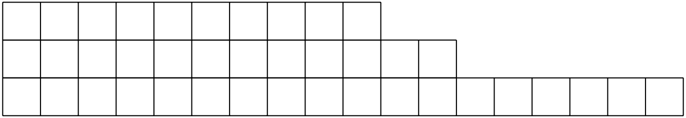
The chain C is periodic of period 2. As $d \to \infty$, C_d approaches alternation between vectors $\mathbf{p}$ and $\mathbf{q} \in S$, where

$$\mathbf{p}_i = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{q}_i = \begin{cases} \frac{\binom{k}{i}}{2^{k-1}} & i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Fix $d > 0$ and pick a Young diagram of size $n - d$ with k rows



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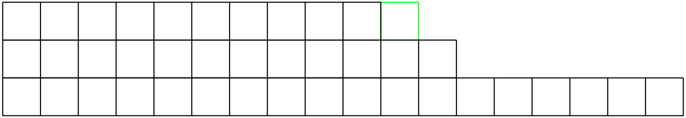

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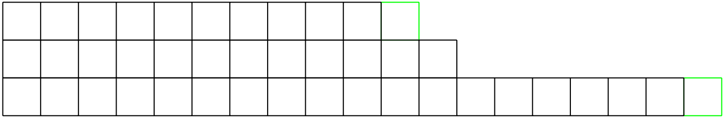

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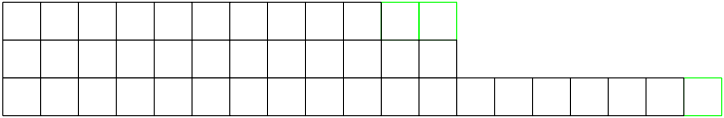

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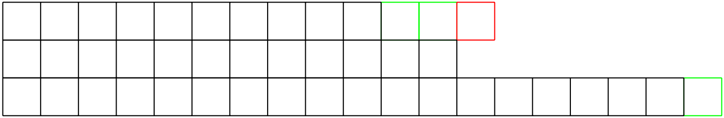

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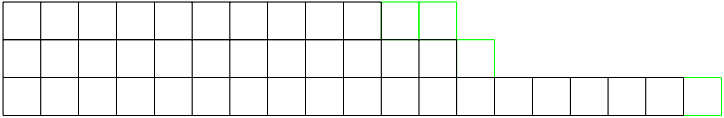

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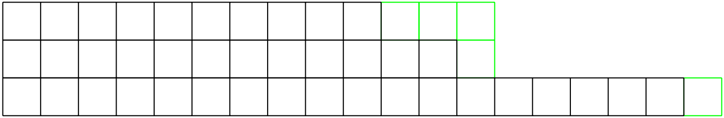

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Weights on Young diagrams λ of size n.

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Two measures on standard Young diagrams:

$$\mu_1 = \frac{f_\lambda}{\sum_\gamma f_\gamma} \qquad \text{and} \qquad \mu_2 = \frac{f_\lambda s_\lambda^d}{\sum_\gamma f_\gamma s_\gamma^d}.$$

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By Matsumoto $\mathbb{P}(s_\lambda^d < k^d) \to 0$.

Thanks!