Staircases, *dominoes and leaves*:
Bounds on gr(Av(1324))

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(Based on joint work with **Robert Brignall**, **Andrew Elvey Price** & **Jay Pantone**)

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29th June 2017

Bounds on the growth rate of 1324-avoiders

$\mathsf{Av}(1324)$ is the only unenumerated class avoiding a pattern of length 4.

$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}_n|^{1/n}$$

	Lower	Upper
2004: Bóna		288
2005: Bóna	9	
2006: Albert et al.	9.47	
2012: Claesson, Jelínek & Steingrímsson†		16
2014: Bóna		13.93
2015: Bóna		13.74
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2015: Conway & Guttmann estimate $\mathrm{gr}(\mathsf{Av}(1324)) \approx 11.60 \pm 0.01$

†An upper bound of 13.002 follows from an unproven conjecture.

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This work	**10.27**	**13.5**

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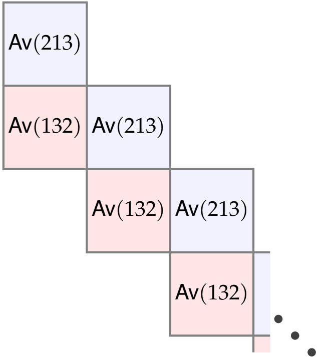
The staircase

The infinite decreasing $(\mathsf{Av}(213), \mathsf{Av}(132))$ staircase:

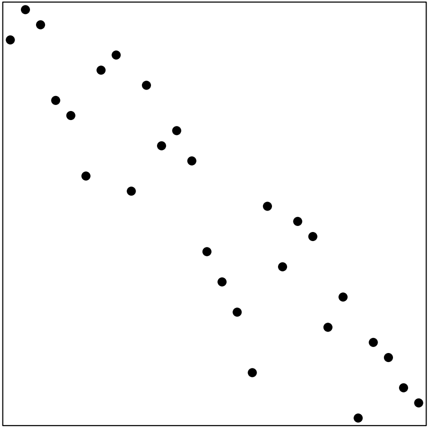
$\mathcal{S} \quad =$



Proposition

$$\mathsf{Av}(1324) \subset \mathcal{S}$$

Gridding a 1324-avoider



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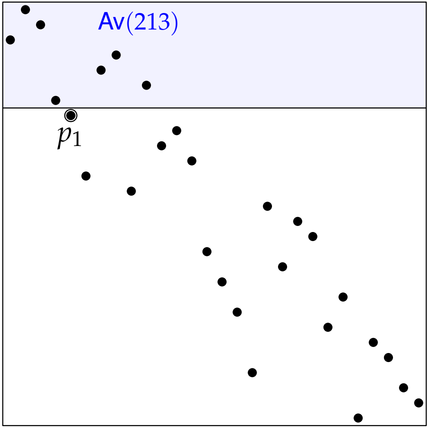
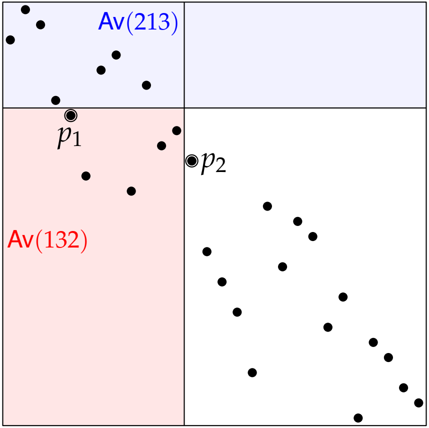


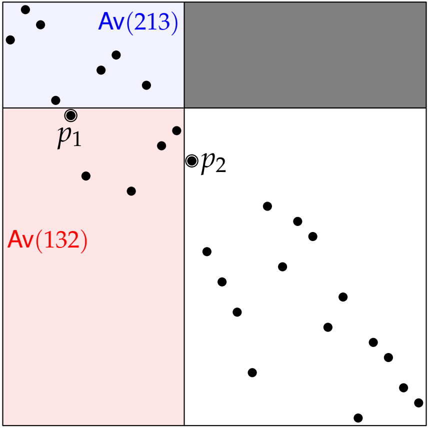
- p_1 uppermost 1 in a 213

Gridding a 1324-avoider



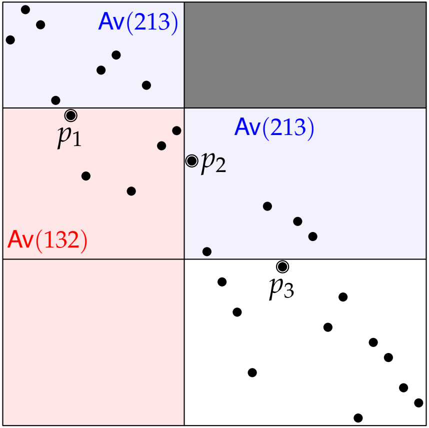
- p_1 uppermost 1 in a 213
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider

Gridding a 1324-avoider



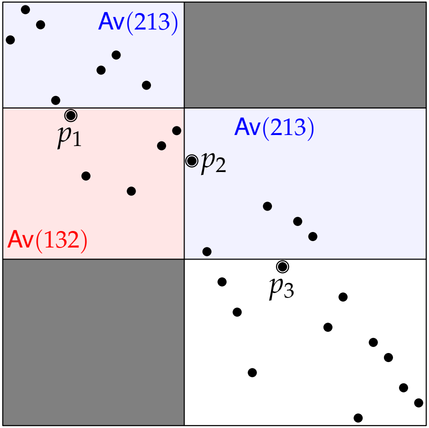
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider
- No points above p_1 and to the right of p_2

Gridding a 1324-avoider



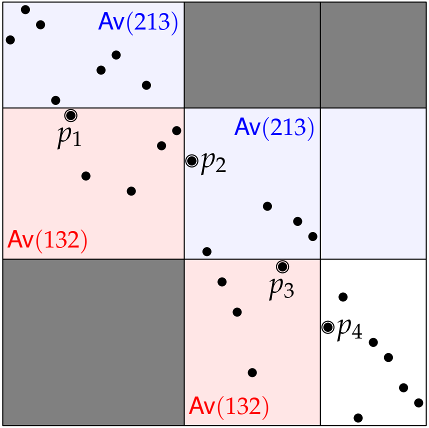
- p_2 leftmost 2 in a 132 consisting of points below p_1 divider
- p_3 uppermost 1 in a 213 consisting of points to right of p_2 divider

Gridding a 1324-avoider



- p_3 uppermost 1 in a 213 consisting of points to right of p_2 divider
- No points to the left of p_2 and below p_3

Gridding a 1324-avoider



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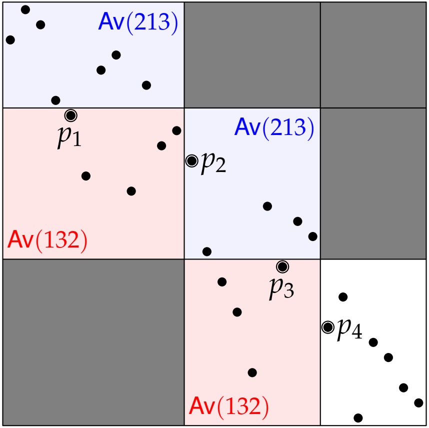

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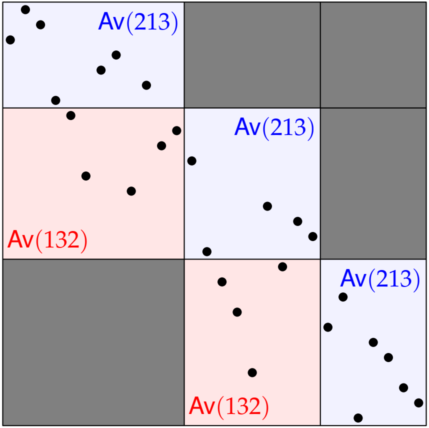
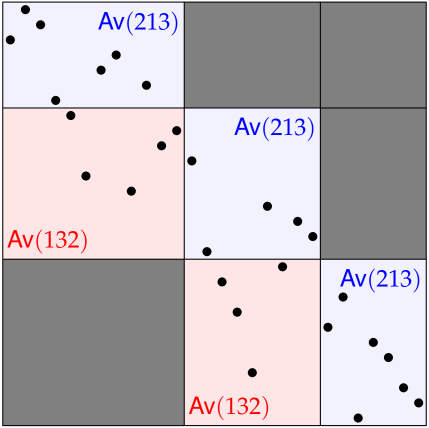
Gridding a 1324-avoider



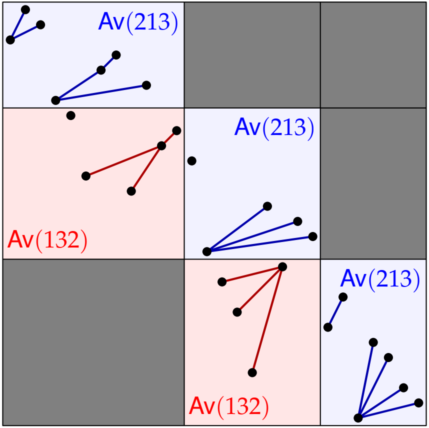
- Terminates after no more than $n/2$ steps.

Gridding a 1324-avoider



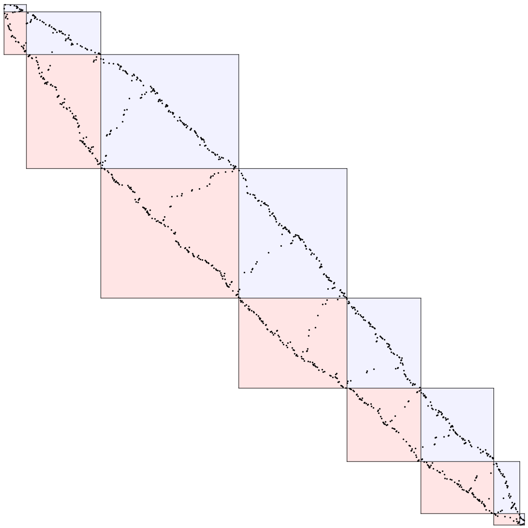
- Terminates after no more than $n/2$ steps.
- Gridding is greedy: each cell contains as many points as possible.

Gridding a 1324-avoider



- Hasse graph of Av(213) is *skew-decomposable* forest of up-trees.
- Hasse graph of Av(132) is *skew-decomposable* forest of down-trees.

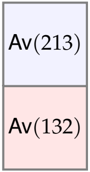
The greedy gridding of a large 1324-avoider



Data provided by Einar Steingrímsson.

Dominoes

A domino is a *gridded permutation* in



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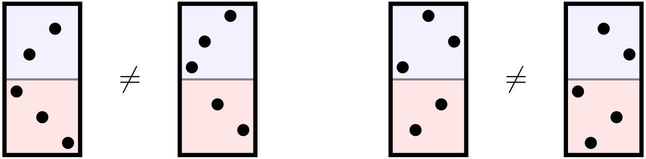

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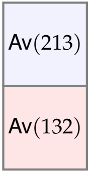

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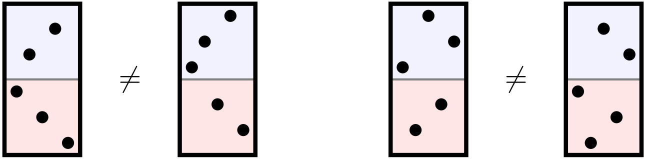
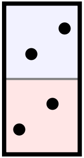


Important:  $\notin \mathcal{D}$ ($\mathcal{D}$ = the set of dominoes)

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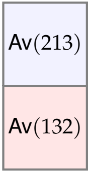

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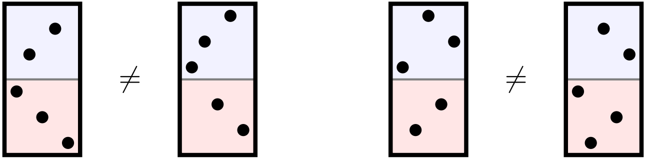
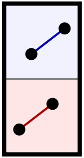


Important:  $\notin \mathcal{D}$ ($\mathcal{D}$ = the set of dominoes)

Dominoes

Theorem

The number of n-point dominoes is $\dfrac{2(3n+3)!}{(n+2)!(2n+3)!}$. $\boxed{\mathrm{gr}(\mathcal{D}) = 27/4.}$

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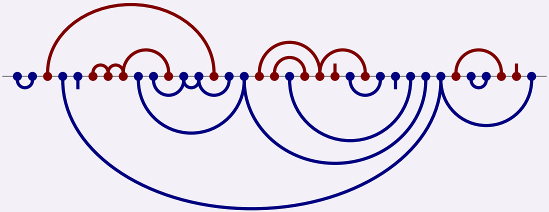
Proof.

Bijection between dominoes and certain arch configurations.



- Functional equation solved using iterated discriminants.

□

Dominoes

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Definition

Jay (*v. tr.*) To confirm a conjectural enumeration by asking Jay Pantone.
Example: *"I doubt that this can be Jayed."*
Jayable (*adj.*)
Example: *"Perhaps this sequence is Jayable."*

Dominoes

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A familiar sequence

Dominoes are equinumerous to

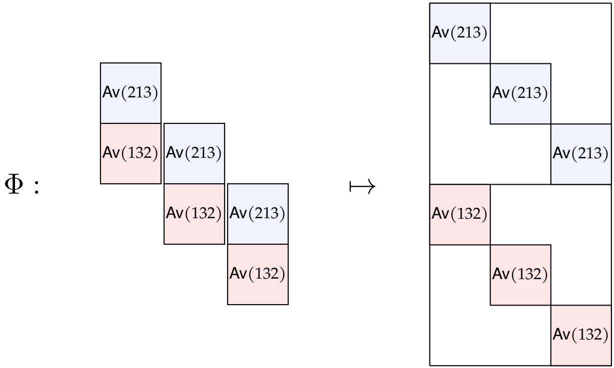
- West-2-stack-sortable permutations
- Rooted nonseparable planar maps

Open problem

Find a bijection between dominoes and another combinatorial structure.

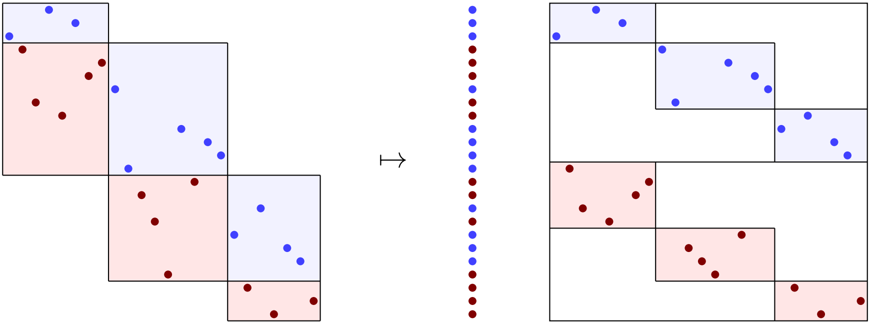
Mapping a 1324-avoider to a domino

- Greedy grid the permutation.
- Interpret the gridded permutation as a *sequence of dominoes*.
- Use Φ to construct a large domino, splitting the small dominoes.
- Φ is not injective; it discards vertical interleaving information.



Upper bound

$$\Psi : \mathsf{Av}_n(1324) \to (\color{blue}{\bullet}, \color{darkred}{\bullet})^n \times \mathcal{D}_n$$



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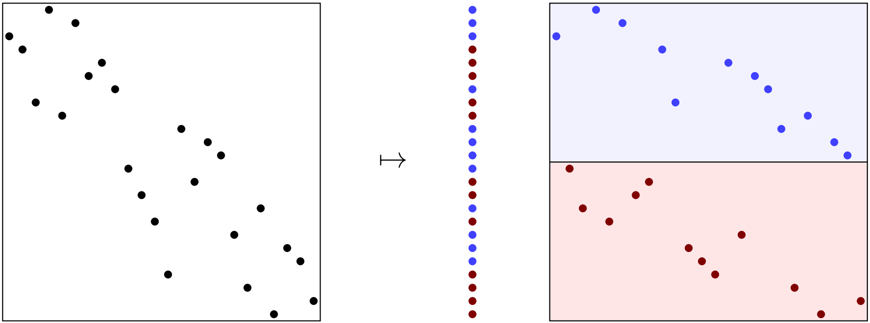

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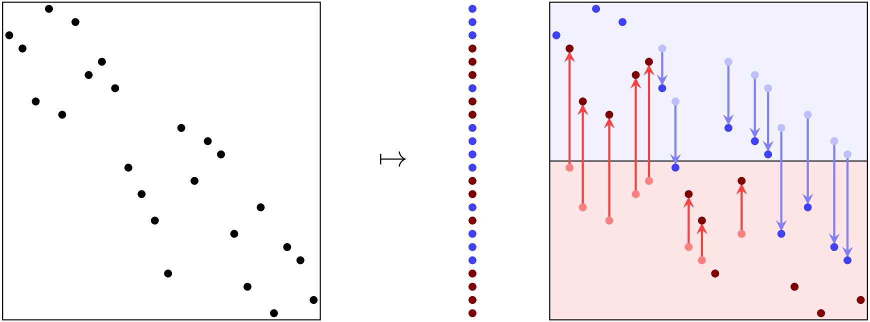

- The vertical interleaving can be recovered from the ●● sequence.

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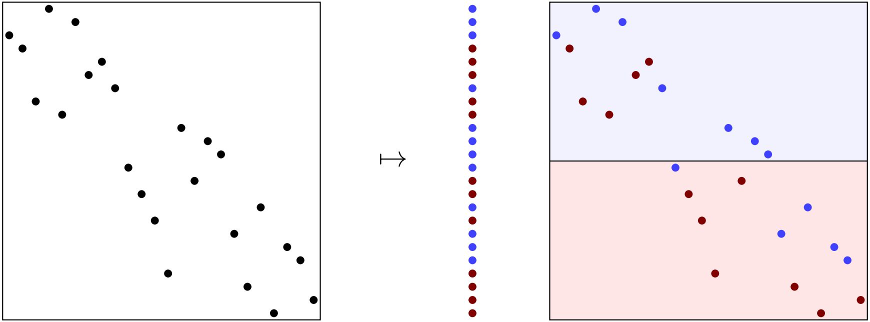
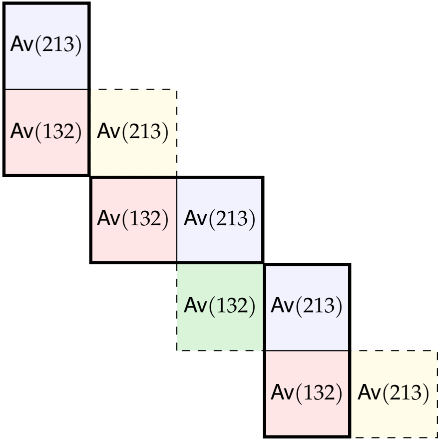


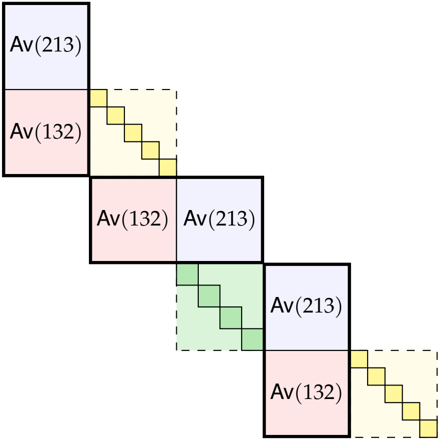
- The vertical interleaving can be recovered from the •• sequence.
- Ψ is an injection. $\boxed{\mathrm{gr}(\mathsf{Av}(1324)) \leqslant 2 \times 27/4 = 13.5}$

Lower bound (1)



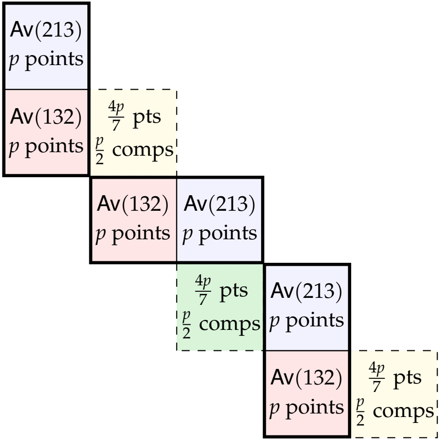
- Decompose staircase into dominoes and *connecting cells*.

Lower bound (1)



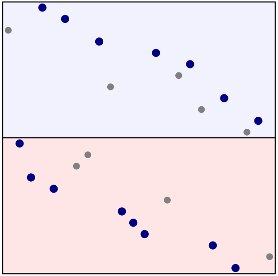
- To avoid 1324, position blue/red points *between* yellow/green skew indecomposable components.

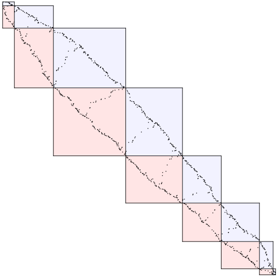
Lower bound (1)



- Optimal values yield $\boxed{\mathrm{gr}(\mathsf{Av}(1324)) \geqslant 81/8 = 10.125}$

Leaves of a domino





Leaves

- left-to-right minima of lower cell
- right-to-left maxima of upper cell

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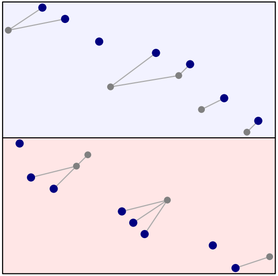 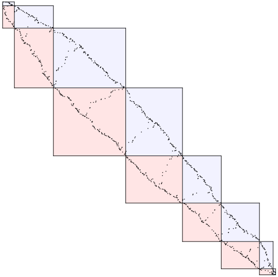

Leaves

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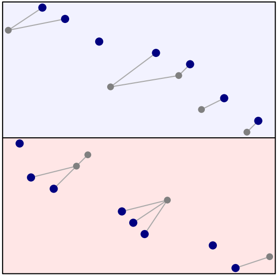
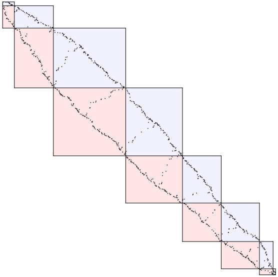
- leaves of trees in Hasse graphs

Leaves of a domino





Leaves

- left-to-right minima of lower cell
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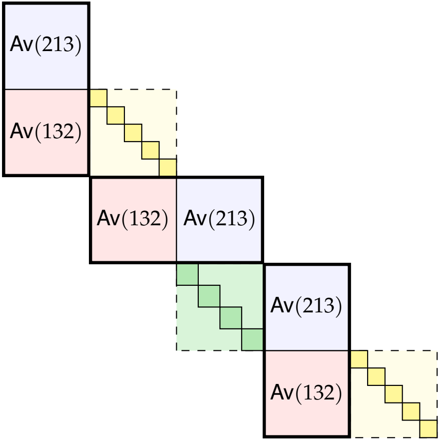
Theorem

The expected number of leaves in an n-point domino is $5n/9$.

Better control of the interleaving



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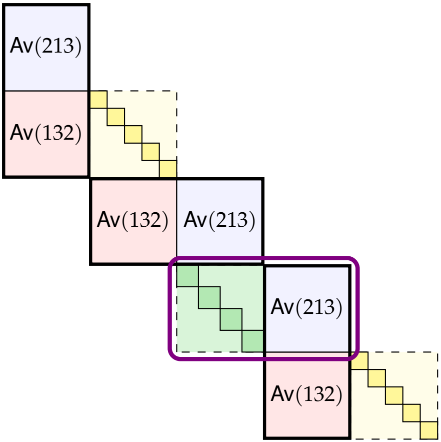

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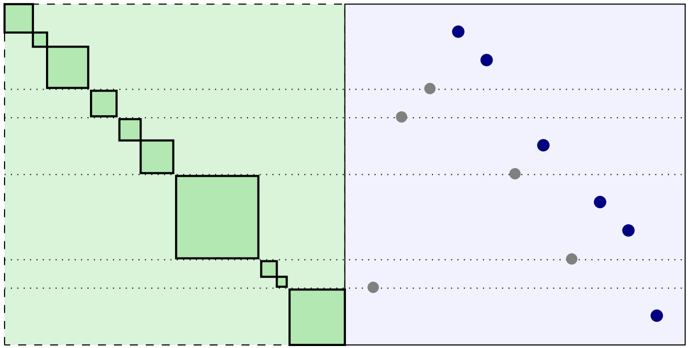
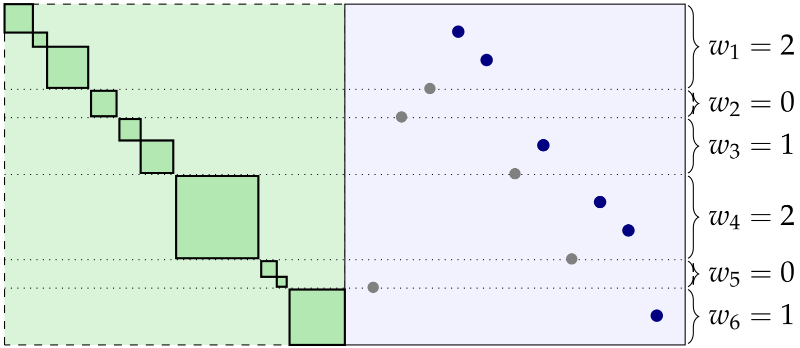


- If every non-leaf occurs *between* yellow/green components, leaves can be *arbitrarily interleaved* without creating a 1324.

Better control of the interleaving



- The non-leaves split a cell into a sequence of strips.
- w_i: width of ith strip (number of leaves in strip)

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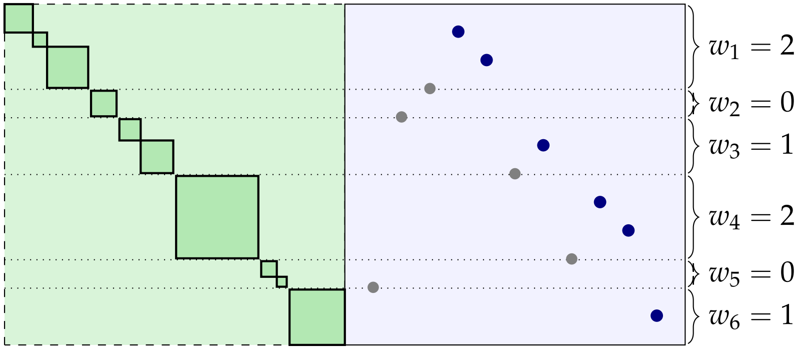

- w_i: width of ith strip (number of leaves in strip)

Theorem

The expected number of empty strips in an n-point domino is $5n/27$.

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- Which strip widths *minimise* the number of ways of interleaving?

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The worst case for interleaving is when the leaves are distributed as evenly as possibly between the strips.

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- $^5/_{12}$ of the strips are empty
- no strips have 1 leaf
- $^1/_2$ of the strips have 2 leaves
- $^1/_{12}$ of the strips have 3 leaves

Lemma

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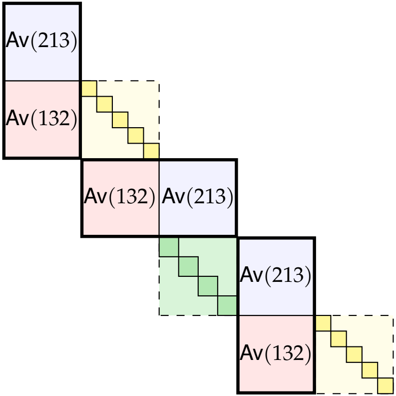
Lower bound (2)



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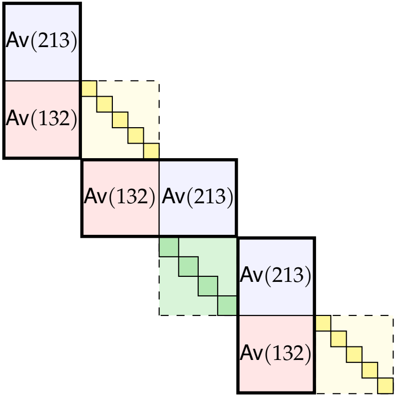

- $\boxed{\mathrm{gr}(\mathsf{Av}(1324)) \geqslant 10.27101292824530\ldots}$

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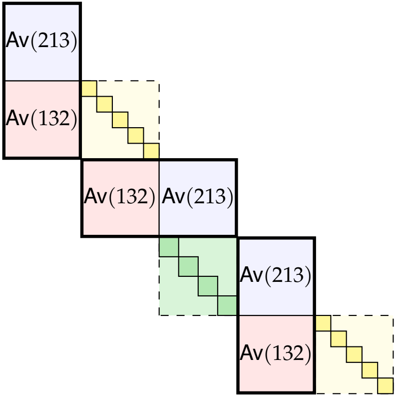

- $\boxed{\mathrm{gr}(\mathsf{Av}(1324)) \geqslant 10.27101292824530\ldots}$
- This value is a root of a polynomial of degree 104, whose smallest coefficient has 86 digits.

Questions for the future

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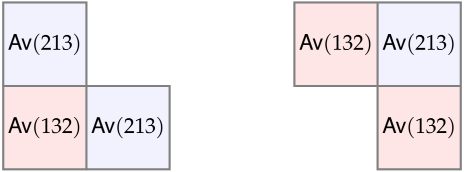
- Trominoes



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- Improvement on the (crude) upper bound
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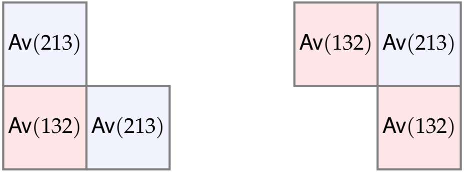
- Trominoes
- "Turning the corner" seems to require new ideas



Thanks!