Patterns in Random Permutations

Chaim Even-Zohar

UC DAVIS
UNIVERSITY OF CALIFORNIA

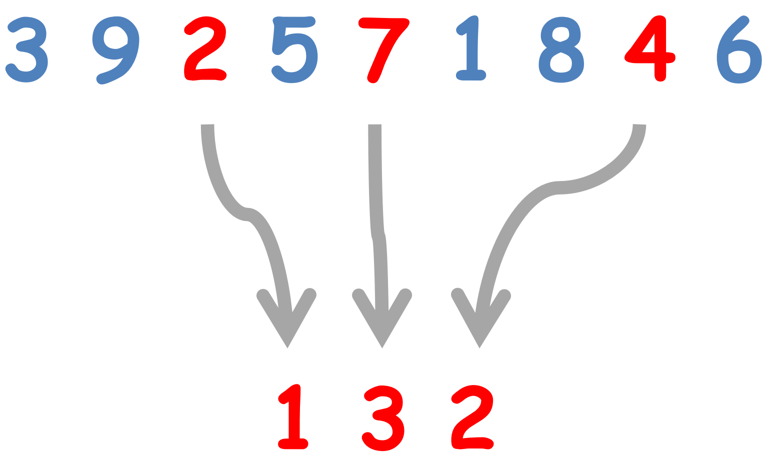
Permutation Patterns

3 9 2 5 7 1 8 4 6

permutation

$\pi \in S_n$

Permutation Patterns



permutation

$\pi \in S_n$

pattern

$\sigma \in S_k$

Pattern Densities

The **density** of $\sigma \in S_k$ in $\pi \in S_n$

$$P_{\sigma}(\pi) = \frac{\#\,\{\sigma \;\; \text{occurs} \;\; \text{in} \;\; \pi\}}{\binom{n}{k}}$$

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The **k-profile** of π

$$\boldsymbol{P}_k(\pi) = (P_\sigma(\pi))_{\sigma \in S_k} \in \mathbb{R}^{k!}$$

The k-Profile

Example $\pi = 4\ 5\ 2\ 1\ 3\ 6$

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$$\boldsymbol{P}_3(\pi) = \begin{bmatrix} P_{123}(\pi) \\ P_{132}(\pi) \\ P_{213}(\pi) \\ P_{231}(\pi) \\ P_{312}(\pi) \\ P_{321}(\pi) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \\ 3 \\ 4 \\ 2 \end{bmatrix} \Big/ \binom{6}{3} = \begin{bmatrix} 0.15 \\ 0 \\ 0.4 \\ 0.15 \\ 0.2 \\ 0.1 \end{bmatrix}$$

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$$P_k(\pi) \cdot 1 = \sum_\sigma P_\sigma(\pi) = 1$$

Randomness

Sample π **at random**

$$\mathbb{P}[\pi = \pi_0] = \frac{1}{n!} \qquad \forall\, \pi_0 \in S_n$$

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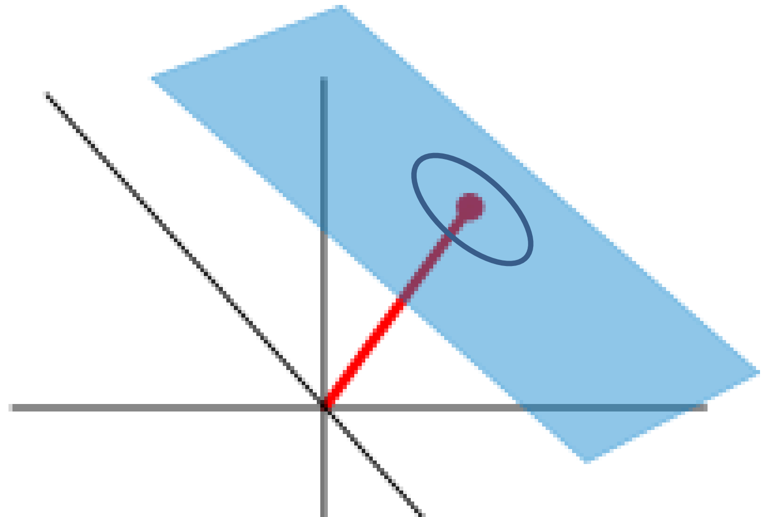
What is the distribution of the **random k-profile** P_{kn}?

Law of Large Numbers

Theorem

In probability,

$$\boldsymbol{P}_{kn} \xrightarrow{n \to \infty} \boldsymbol{U}_k = \begin{pmatrix} 1/k! \\ \vdots \\ 1/k! \end{pmatrix}$$



Law of Large Numbers

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- What is the magnitude of $(\boldsymbol{P}_{kn} - \boldsymbol{U}_k)$?
- What's its direction ?

Central Limit Theorem

Theorem [Janson, Nakamura, Zeilberger '15]

The random vector

$$\sqrt{n}\,(\boldsymbol{P}_{kn} - \boldsymbol{U}_k)$$

weakly converges to

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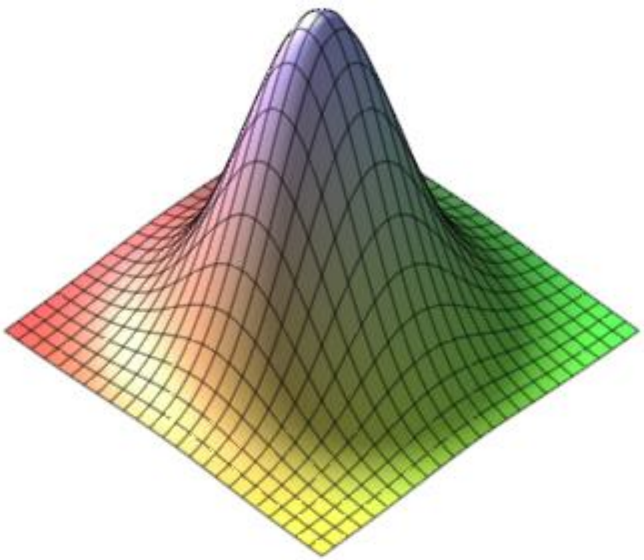


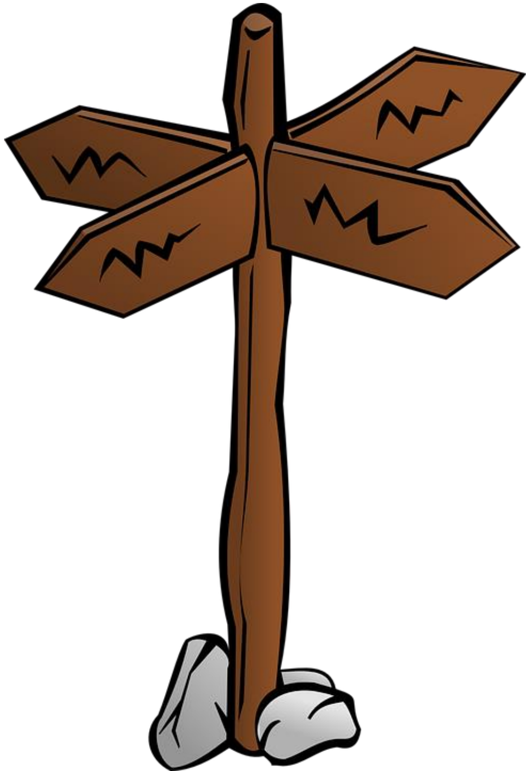
weakly converges to

a multivariate normal distribution, supported on a **$(k-1)^2$**-dimensional subspace.

Components of the Profile

- Projection along U_k
  - constant

$\mathbb{R}^{\boldsymbol{k}}!$



Components of the Profile

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  - Order $1/\sqrt{n}$
  - Asymptotically normal

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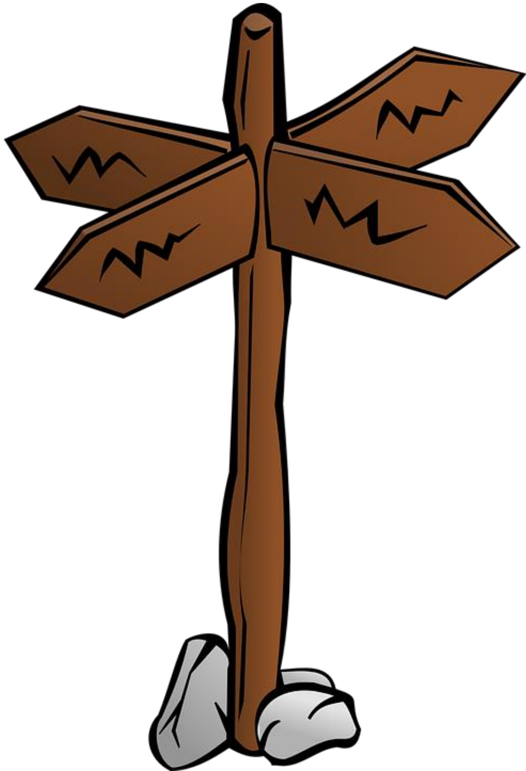

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  - Smaller order

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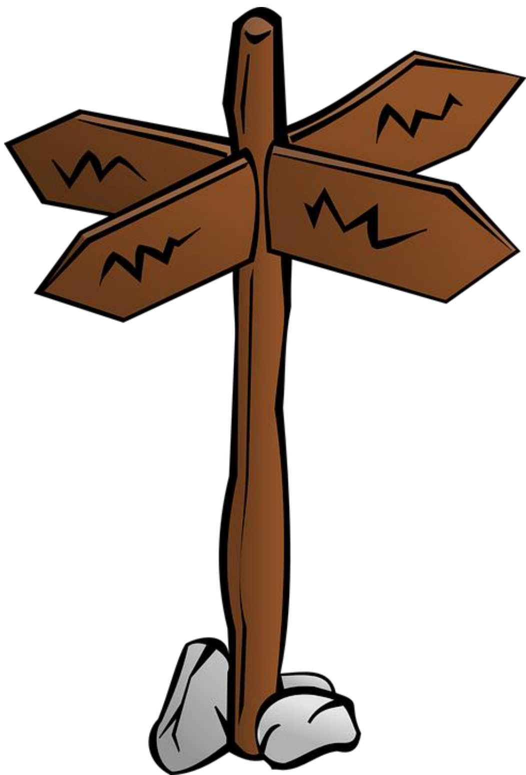

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- Projection along U_k
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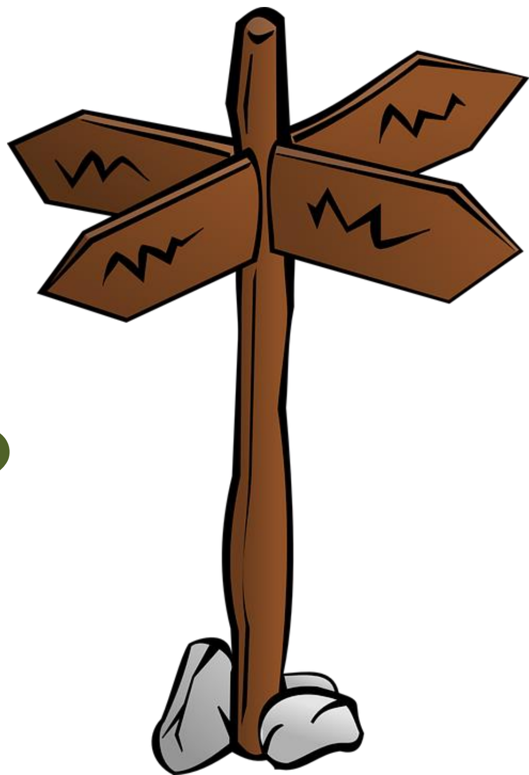
  - Smaller order
  - Which directions, limits

?

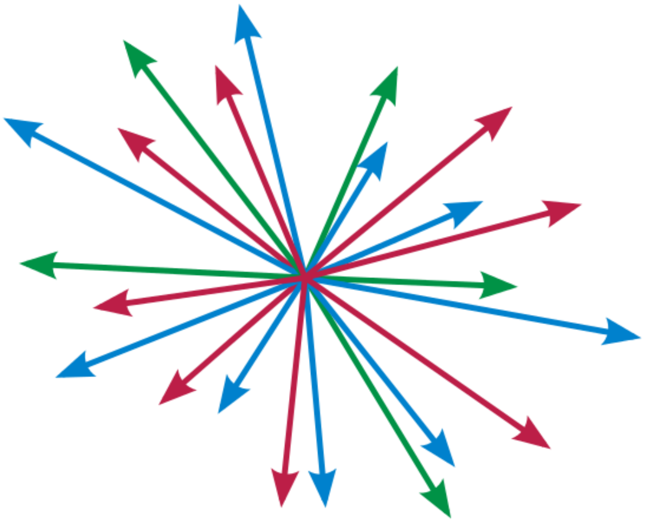
$\mathbb{R}^{k!}$



Decomposition TBD

Pairwise orthogonal

$$\mathbb{R}^{k!} = V_0 \oplus V_1 \oplus \ldots \oplus V_{k-1}$$



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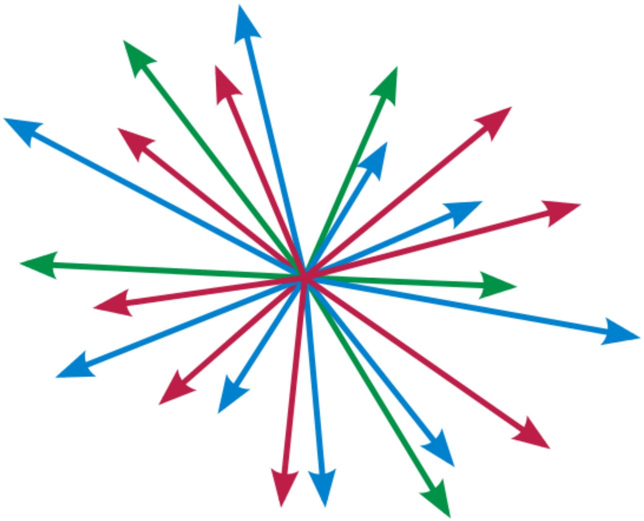
$$\Pi_r : \mathbb{R}^{k!} \longrightarrow V_r$$



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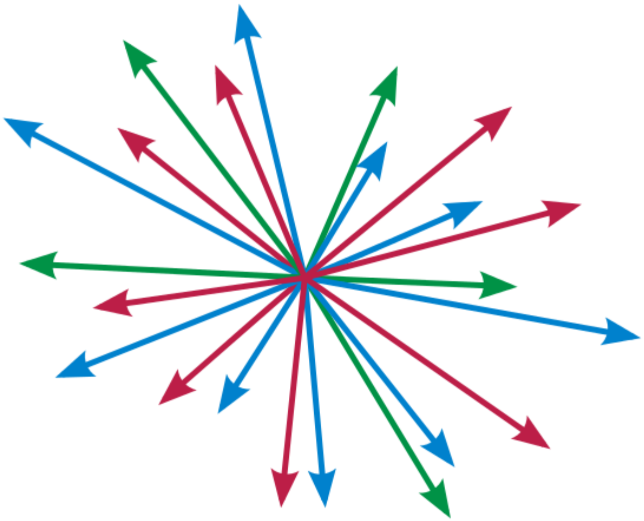

* w.r.t. $\langle u,v \rangle = \sum_\sigma u_\sigma v_\sigma$

Theorem [E]

- For every $r < k$

$$n^{r/2}\, \mathbf{E}\left[\|\pi_r P_{kn}\|\right] \xrightarrow{n \to \infty} \sigma_{kr}$$

for some $0 < \sigma_{kr} < \infty$.

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- For every $r < s < k$

$$\mathbf{E}\left[(n^{r/2}\pi_r P_{kn})\,(n^{s/2}\pi_s P_{kn})^T \right] \xrightarrow{n \to \infty} 0$$

Case k=3

	V_0	V_1				V_2
123	1	1	0	0	1	1
132	1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	-1
213	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	-1
231	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	1
312	1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	1
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312	1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	1
321	1	1	0	0	-1	-1
order	**1**	**$1/\sqrt{n}$**				**$1/n$**

Group Representations

- d-dimensional **representation** of G

$$R : G \longrightarrow GL(\mathbb{R}^d)$$

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- R and R' are **similar** if for some T,

$$R'(g) = T^{-1} \circ R(g) \circ T$$

$\forall\, g \in G$

Group Representations

- d-dimensional **representation** of G

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- R and R' are **similar** if for some T,

$$R'(g) = T^{-1} \circ R(g) \circ T \qquad \forall\, g \in G$$

- R is **simple** if for no proper $V \subset \mathbb{R}^d$

$$R(g)\, V = V \qquad \forall\, g \in G$$

Simple Representations of S_k

$$R^{\lambda} : S_k \longrightarrow \mathbb{R}^{d_\lambda \times d_\lambda}$$

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up to similarity, correspond to **partitions** $\lambda \vdash k$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = k$$
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$$

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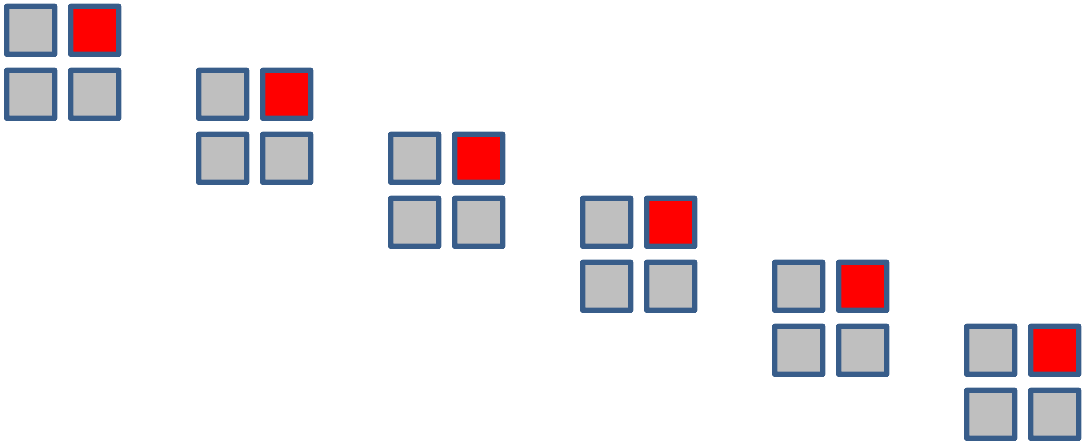
	☺	☺☺	☺	☺
			☺	☺
				☺
4	**3+1**	**2+2**	**2+1+1**	**1+1+1+1**


THE MATRIX
elements

The Matrix Elements

of $\mathbf{R}^{\lambda}$ are the **k!**-dim vectors

$$\boldsymbol{R}_{ij}^{\lambda} = \left(R_{ij}^{\lambda}(\sigma) \right)_{\sigma \in S_k} \qquad 1 \leq i, j \leq d_{\lambda}$$



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$$\boldsymbol{R}_{ij}^{\lambda} = \left(R_{ij}^{\lambda}(\sigma) \right)_{\sigma \in S_k} \qquad 1 \le i,j \le d_\lambda$$

Let

$$\mathrm{V_r} = \mathrm{span} \left\{ \boldsymbol{R}_{ij}^{\lambda} \;\middle|\; \begin{matrix} \lambda \vdash k \\ \lambda_1 = k - r \\ 1 \le i,j \le d_\lambda \end{matrix} \right\}$$

			
123	[1]	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	[1]
132	[1]	$\begin{bmatrix} -\,^{1}/_{2} & ^{\sqrt{3}}/_{2} \\ ^{\sqrt{3}}/_{2} & ^{1}/_{2} \end{bmatrix}$	[-1]
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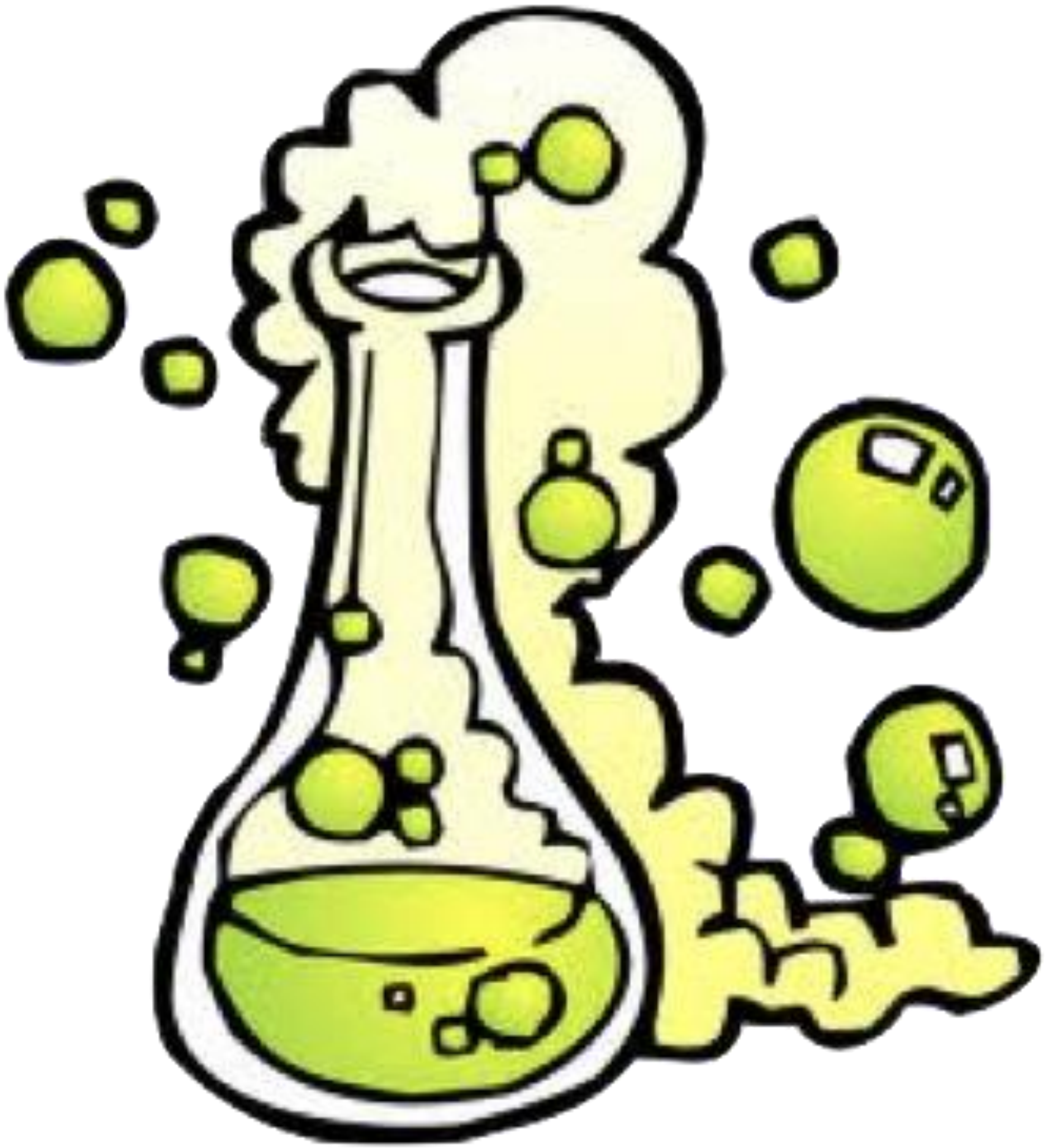
Theorem for k≤6 [E]

There're unitary representations of S_k having

$$E\left[(U_R P_{kn})\,(U_R P_{kn})^T\right] \xrightarrow{n\to\infty} \Sigma$$

is diagonal with positive entries, where

$$U_R\, v = \left(n^{\frac{k-\lambda_1}{2}} \langle \widehat{R}_{ij}^{\lambda}, v\rangle\right)_{\lambda\vdash k,\ 1\le i,j\le d_\lambda}$$



Applications (1)

Kendall's τ / inversion number

$$\tau(\pi) = \langle \boldsymbol{R}_{11}^{1+1}, \boldsymbol{P}_2(\pi) \rangle = P_{12}(\pi) - P_{21}(\pi)$$

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Applications (1)

Kendall's τ / inversion number

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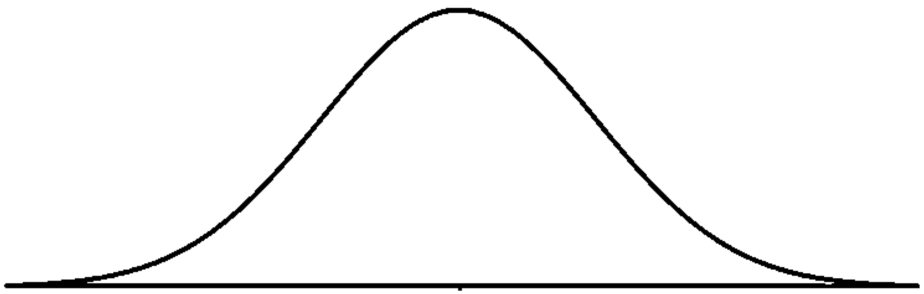
- Order $1/\sqrt{n}$ ☺
- Asymptotically normal ☺



Applications (2)

Spearman's ρ

$$\rho = P_{123} + P_{132} + P_{213} - P_{231} - P_{312} - P_{321}$$

$$\propto \left\langle \boldsymbol{R}_{11}^{2+1} - \frac{1}{4}\boldsymbol{R}_{11}^{1+1+1},\ \boldsymbol{P}_3 \right\rangle$$

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☺

- Order **$1/\sqrt{n}$**
- Asymptotically normal

Applications (3)

Fisher-Lee / Gepner Statistics

$$\Delta = P_{123} + P_{231} + P_{312} - P_{321} - P_{213} - P_{132}$$

$$= \langle \boldsymbol{R}_{11}^{1+1+1}, \boldsymbol{P}_3 \rangle$$

- Circular rank correlation

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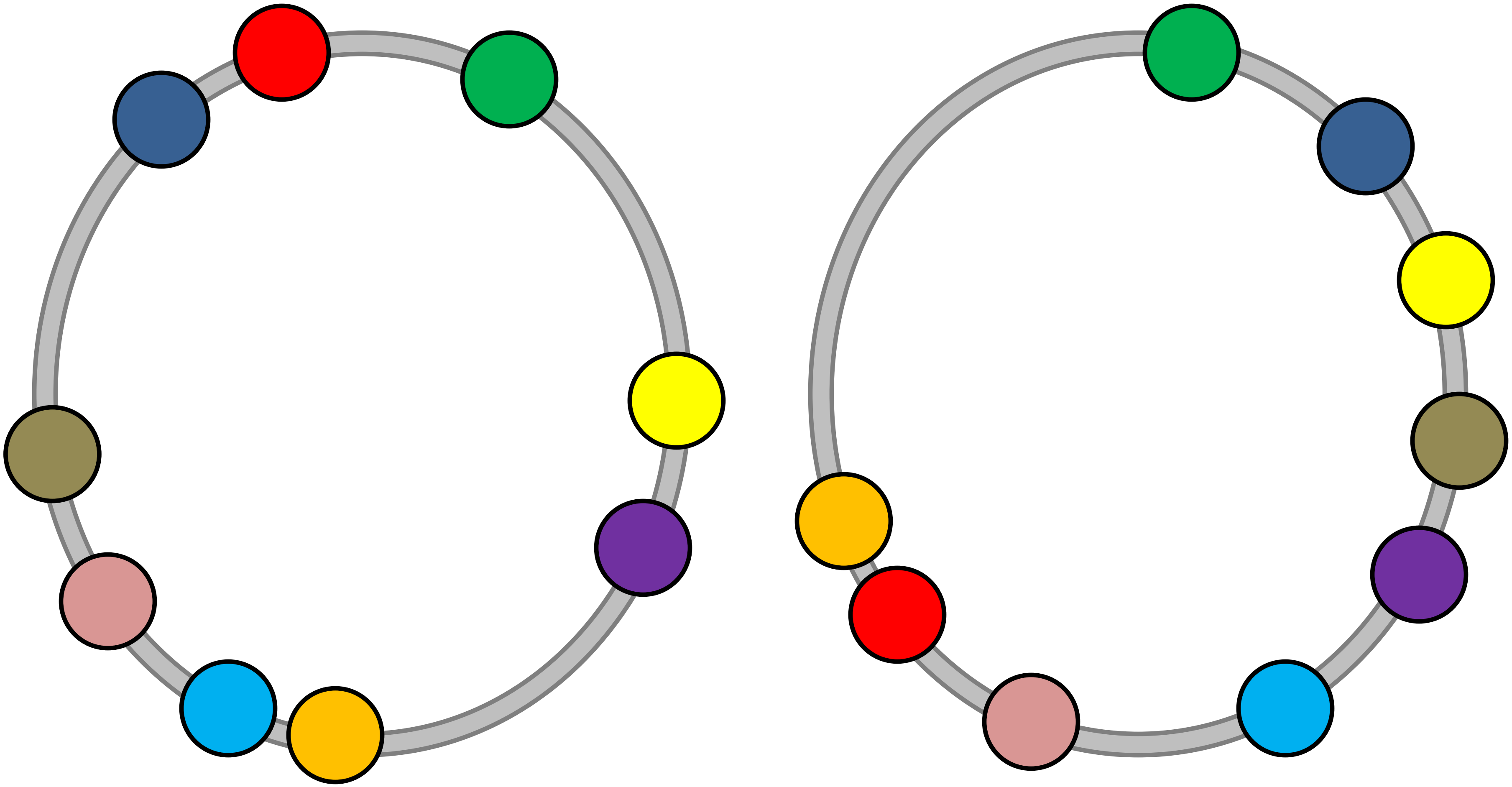

Circular Rank Correlation

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- Circular rank correlation ☺
- Order **1/n** ☺
- Not normal: 2nd order U-statistic ☺

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Fisher-Lee / Gepner Statistics

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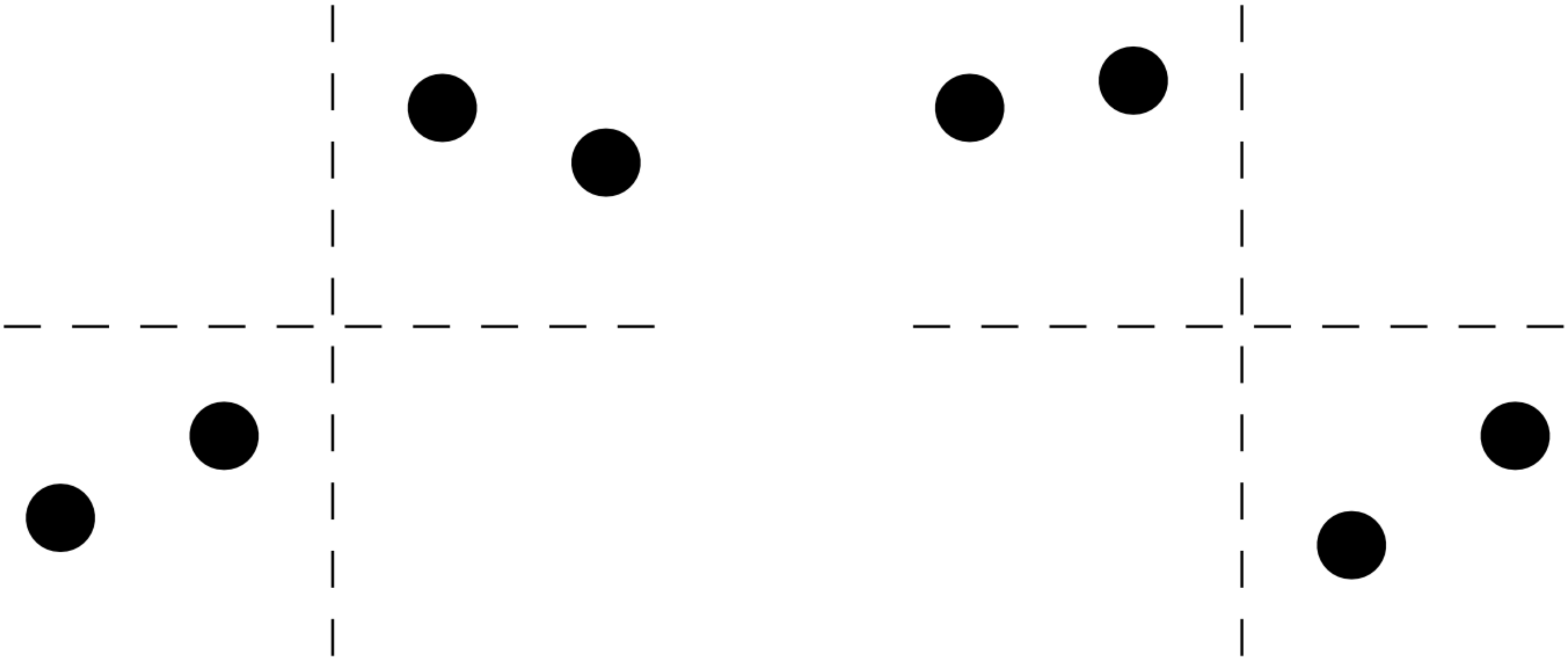
- Circular rank correlation
- Order $1/n$ ☺
- Not normal: 2nd order U-statistic ☺
- $\propto (\tau - \rho)$ ☺

Applications (4)

Two-Sample Independence Tests

$$P_{1234} + P_{1243} + P_{2134} + P_{2143} + P_{3412} + P_{3421} + P_{4312} + P_{4321}$$

- Order $1/n$
- 2nd order U-statistic



Bergsma-Dassios (2010)

Applications (4)

Two-Sample Independence Tests

$$P_{1234} + P_{1243} + P_{2134} + P_{2143} + P_{3412} + P_{3421} + P_{4312} + P_{4321}$$

- Order $1/n$
- 2nd order U-statistic
  - Bergsma-Dassios $\langle R_{11}^{2+2}, P_4 \rangle$ ☺☺☺☺
  - Hoeffding $+ \ldots \langle R_{11}^{2+2+1}, P_5 \rangle$ ☺☺☺☺☺
  - Blum-Kiefer-Rosenblatt $+ \ldots \langle R_{11}^{3+2+1}, P_6 \rangle$ ☺☺☺☺☺☺
  - Král' Pikhurko

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