Prolific permutations and permuted packings

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We will describe permutations of size $n$ both as:

words $\pi(1) \cdots \pi(n) \in S_n$ (to capture the idea of “pattern”), and

plots of points $\{(i, \pi(i)) : 1 \leq i \leq n\} \subset \mathbb{R}^2$ (for our proof)

Example. $31524 \in S_5$
Pattern poset

Write $\sigma \preceq \pi$ if $\pi$ contains a $\sigma$-pattern.

The pattern poset $\mathcal{P}$ is $\bigcup_{k \geq 1} S_k$, ordered by $\preceq$. 

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The poset is complicated, as we already know and appreciate.

Principal order ideals have many forms, even from the same rank.

In particular, note the different widths in these pictures.
The definition

\[ \pi \in S_n \text{ is } k\text{-prolific if } \left| \{ \sigma \in S_{n-k} : \sigma \preceq \pi \} \right| = \binom{n}{k}. \]

I.e., if each \((n - k)\)-subset of letters in \(\pi\) forms a distinct pattern.

I.e., if \(\pi\) has maximally many descendants \(k\) generations down in \(\mathcal{P}\).
Example. 1234 is not 1- or 2-prolific.

Example. 1243 is not 1- or 2-prolific.

Example. 2413 is 1-prolific, but not 2-prolific.
The questions

Do $k$-prolific permutations exist for all $k$?

If $w \in S_n$ is $k$-prolific, how big must $n$ be, as a function of $k$?

How common are prolific permutations?
Some answers

**Thm.** $k$-prolific permutations exist for every $k$.

**Thm.** For any $n \geq k^2/2 + 2k + 1$, there exists a $k$-prolific permutation in $S_n$.

In fact, this is strict: there is no $k$-prolific permutation in $S_n$ when $n < k^2/2 + 2k + 1$. 
The **breadth** of $\pi$ is

$$br(\pi) = \min_{i,j} \{|i - j| + |\pi(i) - \pi(j)|\}$$

This is the minimum **taxicab distance** in the plot of $\pi$. Certainly $br(\pi) \geq 2$.

**Example.** $br(31524) = 3$ and $br(274915836) = 4$. 
Key to our proof

**Lem.** $\pi$ is $k$-prolific iff $br(\pi) \geq k + 2$.

**Example.** Consider $\pi = 31524$, which has breadth 3.

$\pi$ is 1-prolific, with $\binom{5}{1}$ children: 1423, 2413, 3124, 2143, 3142.

$\pi$ is not 2-prolific because it has only $5 < \binom{5}{2}$ grandchildren in $P$: 123, 132, 213, 231, and 312.

**Example.** $274915836$ (breadth 4) is 2-prolific.
Prop. Deleting a single entry from a permutation decreases breath by at most 1.

Proof: If $\text{br}(\sigma) < \text{br}(\pi)$, then $\text{br}(\pi) = |i - j| + |\pi(i) - \pi(j)|$, and $\sigma$ was obtained by deleting a point with $x$-coordinate between $i$ and $j$, or $y$-coordinate between $\pi(i)$ and $\pi(j)$.

The deleted point could not have satisfied both of those requirements, or $\text{br}(\pi)$ would be smaller than had been claimed.
Lem. If $\pi$ is $k$-prolific, then $\text{br}(\pi) \geq k + 2$.

Proof: If $|i - j| + |\pi(i) - \pi(j)| \leq k + 1$, then we have:

![Diagram showing shaded regions and points (i, $\pi(i)$) and (j, $\pi(j)$).]

where the number of points in the shaded regions is at most $k - 1$.

Deleting $\{\text{shaded points}\} \cup \{(i, \pi(i))\}$ results in the same perm as deleting $\{\text{shaded points}\} \cup \{(i, \pi(i))\}$, so $\pi$ is not $k$-prolific.
Lem. If \( \text{br}(\pi) \geq k + 2 \), then \( \pi \) is \( k \)-prolific.

This is notably more complicated.

Induct on \( k \), and use **chain graphs** to describe how two different occurrences of the same pattern might be contained in \( \pi \).

Chain graphs have a lot of structure: if there are two ways to get the same pattern in \( S_{n-k} \), then two points in the plot must have taxicab distance \( < k + 2 \), meaning \( \text{br}(\pi) < k + 2 \).

Next up: What does breadth mean for constructing \( k \)-prolific permutations?
A packing $\Pi$ of $n$ translates of a tile $T$ is a **permuted packing** if $\Pi = \{T + (i, \pi(i))\}$ for some $\pi \in S_n$.

Let $D_k$ be a diamond whose diagonal has length $k + 2$.

$k$-prolific perms are in **bijection** with permuted packings of $D_k$. 
Examples

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Construction bounds

Let \( \text{minprol}(k) \) be the minimum value of \( n \) for which there exists a \( k \)-prolific permutation in \( S_n \).

We get size requirements for \( k \)-prolific perms from area restrictions. This gives us a lower bound: \( \lceil k^2/2 + 2k + 1 \rceil \leq \text{minprol}(k) \)

We get the upper bound by explicit construction.

**Thm.** \( \text{minprol}(k) = \lceil k^2/2 + 2k + 1 \rceil \)

In fact, we can “grow” the constructed \( k \)-prolific permutations, so –

**Thm.** There exist \( k \)-prolific perms in \( S_n \) for all \( n \geq \lceil k^2/2 + 2k + 1 \rceil \).
Additional questions and progress

For a given $k > 1$, how does the number of $k$-prolific permutations of size $n$ grow with $n$?

**Blackburn-Homberger-Winkler:** For large $n$, a random $n$-permutation is $k$-prolific with probability $e^{-k^2-k}$.

How many distinct $k$-prolific permutations exist of minimal size?

**Conjecture:** For odd $k$, the minimal permutation we construct, and its symmetries, are the only ones.

Variations on prolificity?