Prolific permutations and permuted packings

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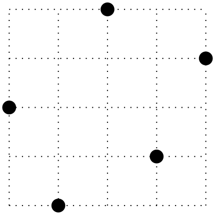
Notation

We will describe permutations of size n both as:

words $\pi(1) \cdots \pi(n) \in S_n$ (to capture the idea of "pattern"), and

plots of points $\boldsymbol{\{(i, \pi(i)) : 1 \leq i \leq n\}} \subset \mathbb{R}^2$ (for our proof)

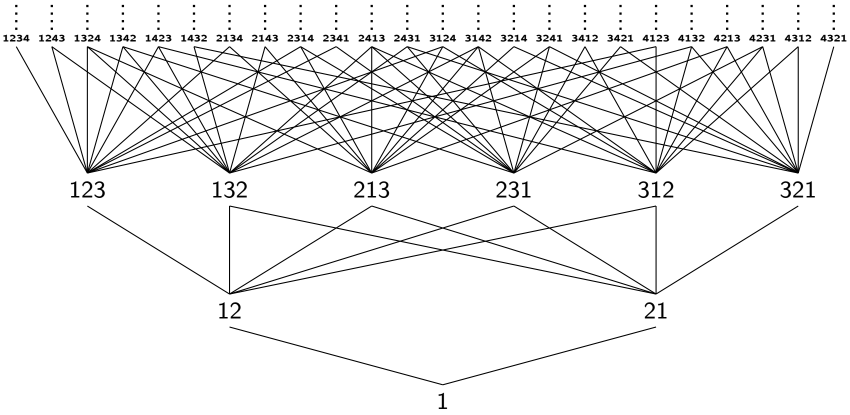
Example. $31524 \in S_5$



Pattern poset

Write $\boldsymbol{\sigma} \preceq \boldsymbol{\pi}$ if π contains a σ-pattern.

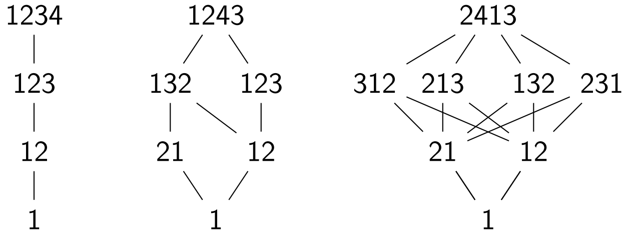
The **pattern poset** $\mathcal{P}$ is $\bigcup_{k\geq 1} S_k$, ordered by $\preceq$.



Principal order ideals

The poset is complicated, as we already know and appreciate.

Principal order ideals have many forms, even from the same rank.



In particular, note the **different widths** in these pictures.

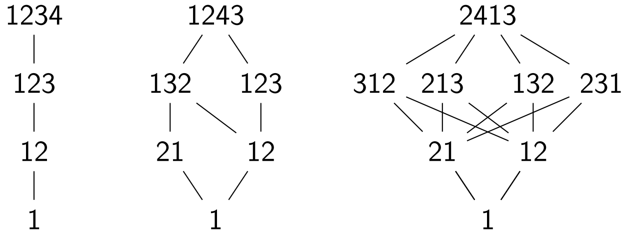
The definition

$\pi \in S_n$ is k-**prolific** if $$\left|\{\sigma \in S_{n-k} : \sigma \preceq \pi\}\right| = \binom{n}{k}.$$

I.e., if each $(n-k)$-subset of letters in π forms a distinct pattern.

I.e., if π has maximally many descendants k generations down in $\mathcal{P}$.

Examples



Example. 1234 is not 1- or 2-prolific.

Example. 1243 is not 1- or 2-prolific.

Example. 2413 is 1-prolific, but not 2-prolific.

The questions

Do k-prolific permutations **exist** for all k?

If $w \in S_n$ is k-prolific, **how big** must n be, as a function of k?

How **common** are prolific permutations?

Some answers

Thm. k-prolific permutations **exist for every** k.

Thm. For any $\boldsymbol{n \geq k^2/2 + 2k + 1}$, there exists a k-prolific permutation in S_n.

In fact, this is strict: there is no k-prolific permutation in S_n when $n < k^2/2 + 2k + 1$.

Tool: breadth

The **breadth** of π is

$$\mathbf{br}(\pi) = \min_{i,j} \left\{ |i-j| + |\pi(i) - \pi(j)| \right\}$$

This is the minimum **taxicab distance** in the plot of π.
Certainly $\mathrm{br}(\pi) \geq 2$.

Example. $\mathrm{br}(31524) = 3$ and $\mathrm{br}(274915836) = 4$.

Key to our proof

Lem. π is k-prolific iff $\mathbf{br}(\pi) \geq \boldsymbol{k} + \mathbf{2}$.

Example. Consider $\pi = 31524$, which has breadth 3.

π is 1-prolific, with $\binom{5}{1}$ children: 1423, 2413, 3124, 2143, 3142.

π is not 2-prolific because it has only $5 < \binom{5}{2}$ grandchildren in $\mathcal{P}$: 123, 132, 213, 231, and 312.

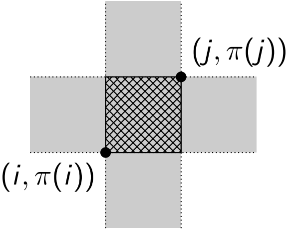
Example. 274915836 (breadth 4) is 2-prolific.

Working toward the lemma

Prop. Deleting a single entry from a permutation decreases breath by at most 1.

Proof: If $\mathrm{br}(\sigma) < \mathrm{br}(\pi)$, then $\mathrm{br}(\pi) = |i - j| + |\pi(i) - \pi(j)|$, and σ was obtained by deleting a point with x-coordinate between i and j, or y-coordinate between $\pi(i)$ and $\pi(j)$.

π:

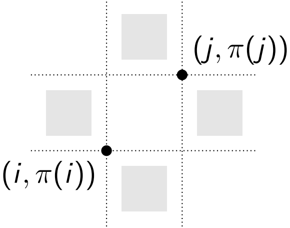


The deleted point could not have satisfied both of those requirements, or $\mathrm{br}(\pi)$ would be smaller than had been claimed.

Proving the lemma, part 1

Lem. If π is k-prolific, then $\mathrm{br}(\pi) \geq k + 2$.

Proof: If $|i - j| + |\pi(i) - \pi(j)| \leq k + 1$, then we have:



where the number of points in the shaded regions is at most $k - 1$.

Deleting $\{\text{shaded points}\} \cup \{(i, \pi(i))\}$ results in the same perm as deleting $\{\text{shaded points}\} \cup \{(i, \pi(i))\}$, so π is not k-prolific.

Proving the lemma, part 2

Lem. If $\mathrm{br}(\pi) \geq k + 2$, then π is k-prolific.

This is notably more complicated.

Induct on k, and use **chain graphs** to describe how two different occurrences of the same pattern might be contained in π.

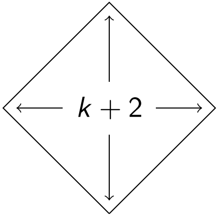
Chain graphs have a lot of structure: if there are two ways to get the same pattern in S_{n-k}, then two points in the plot must have taxicab distance $< k + 2$, meaning $\mathrm{br}(\pi) < k + 2$.

Next up: What does breadth mean for constructing k-prolific permutations?

Permuted packings

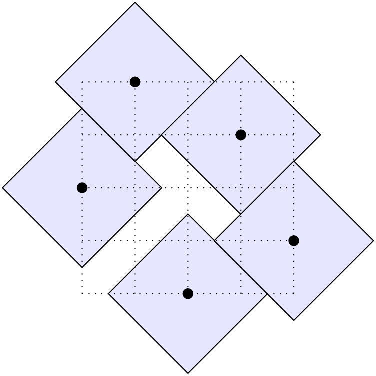
A packing Π of n translates of a tile T is a **permuted packing** if $\Pi = \{T + (i, \pi(i))\}$ for some $\pi \in S_n$.

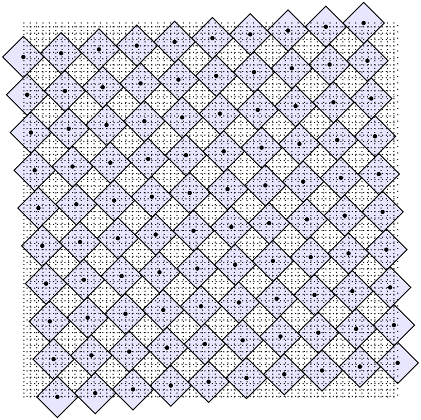
Let $\boldsymbol{D_k}$ be a diamond whose diagonal has length $k + 2$.



k-prolific perms are in **bijection** with permuted packings of D_k.

Examples





Construction bounds

Let **minprol(k)** be the minimum value of n for which there exists a k-prolific permutation in S_n.

We get size requirements for k-prolific perms from area restrictions.

This gives us a lower bound: $\lceil \boldsymbol{k^2/2 + 2k + 1} \rceil \leq$ **minprol(k)**

We get the upper bound by explicit construction.

Thm. minprol$(k) = \lceil k^2/2 + 2k + 1 \rceil$

In fact, we can "grow" the constructed k-prolific permutations, so –

Thm. There exist k-prolific perms in S_n for all $n \geq \lceil \frac{k^2}{2} + 2k + 1 \rceil$.

Additional questions and progress

For a given $k > 1$, how does the number of k-prolific permutations of size n grow with n?

Blackburn-Homberger-Winkler: For large n, a random n-permutation is k-prolific with probability e^{-k^2-k}.

How many distinct k-prolific permutations exist of minimal size?

Conjecture: For odd k, the minimal permutation we construct, and its symmetries, are the only ones.

Variations on prolificity?