Asymptotics of Pattern Classes of Set Partition and Permutation d-tuple Avoidance

Benjamin Gunby

Harvard Department of Mathematics

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This research is joint work with Dömötör Pálvölgyi.

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Paper hopefully on arXiv soon!

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- We will call these sets *blocks* of the partition.
- Since the order of sets is irrelevant, we will order the B_i in increasing order of smallest element, and denote the set partition with slashes between the blocks. (e.g. $\{5, 2, 3\} \cup \{4, 6, 1\}$ is denoted 146/235).

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In other words, π contains π' if and only if we can restrict π to a k-element subset of $[n]$, so that the resulting set partition is order-isomorphic to π.
(Example: $146/235$ contains the partition $12/34$, as we can see by restricting $146/235$ to the set $\{2, 3, 4, 6\}$.)

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3. *There exists $d \in \mathbb{Z}^+$ and constants $c' > c > 0$ such that*

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Question

Which pattern classes fall into which growth rates? That is, which d corresponds to a given $\mathcal{P}$?

Permutability

Definition

Given $n, d \in \mathbb{Z}^+$, and $\sigma_1, \ldots, \sigma_d \in S_n$, we can construct a set partition of $[(d+1)n]$ as follows: there will be n blocks $B_1, \ldots, B_n$, with $B_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, dn + \sigma_d(i)\}$. We call this set partition $[\sigma_1, \ldots, \sigma_d]$.

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Example: $[132, 321] = 149/268/357$

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Let π be a set partition. Then the permutability of π, denoted $pm(\pi)$ is the minimum positive integer d such that π is contained in a set partition of the form $[\sigma_1, \ldots, \sigma_d]$, for some $m \in \mathbb{Z}^+$ and $\sigma_1, \ldots, \sigma_d \in S_m$.

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Definition (Alternate)

The set partition π of $[n]$ has permutability d if and only if $[n]$ can be divided into $d+1$ intervals such that each interval contains at most one element of each block of π, but it cannot be divided into d intervals in that way.

More Specific Main Result

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In this case, d is the smallest permutability among set partitions that do not occur in $\mathcal{P}$ (except when $d = 1$; then this may be 0).

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Example: if $\mathcal{P} = Av(\pi)$, then $d = \mathrm{pm}(\pi)$.

Lower Bound

Theorem

$$c^n n^{\left(1-\frac{1}{d}\right)n} < |\mathcal{P}_n| < c'^n n^{\left(1-\frac{1}{d}\right)n},$$

where d is the smallest permutability not occurring in $\mathcal{P}$.

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$$> \left(\frac{n}{ed}\right)^{\frac{d-1}{d}n},$$

proving the lower bound of the theorem.

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The upper bound is more complicated; we will describe an important lemma.

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If $\mathcal{P}$ is a pattern class not containing some π of permutability d, then $\mathcal{P} \subset Av(\pi) \subset Av([\sigma_1, \ldots, \sigma_d])$ for some permutations $\sigma_1, \ldots, \sigma_d$.

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Thus it suffices to show that $Av_n([\sigma_1, \ldots, \sigma_d]) < c'^n n^{\left(1-\frac{1}{d}\right)n}$ for some c'.

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Definition

A d-permutation hypergraph be the hypergraph corresponding to some set partition of the form $[\sigma_1, \ldots, \sigma_{d-1}]$ *(that is, the image of* $[\sigma_1, \ldots, \sigma_{d-1}]$ *under the bottom map in the diagram above).*

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For example, G contains the hypergraph H on $[4]$ with edges $\{1,3\}$ and $\{2,4\}$ if and only if there exist two *different* edges E_1 and E_2 of G and vertices $v_1, v_1' \in E_1$, $v_2, v_2' \in E_2$ with $v_1 < v_2 < v_1' < v_2'$. (A single 4-vertex edge would not suffice.)

Main Lemma

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Lemma (G., Pálvölgyi)

Let H be a d-permutation hypergraph. Then there exists c such that any ordered hypergraph G on $[n]$ that avoids H has

$$\sum_{E \in E(G)} |E| < cn^{d-1}.$$

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- The case where G is also a d-permutation hypergraph is a theorem of Klazar-Marcus.

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Let $\sigma_1, \ldots, \sigma_d \in S_n$ and $\sigma'_1, \ldots, \sigma'_d \in S_m$ be permutations. Then the d-tuple $(\sigma_1, \ldots, \sigma_d)$ contains (respectively avoids) the d-tuple $(\sigma_1, \ldots, \sigma_d)$ if there exist (respectively do not exist) indices $i_1, \ldots, i_m \in [n]$ with $i_1 < \cdots < i_m$ satisfying the property that for any j, $\sigma_j(i_1) \cdots \sigma_j(i_m)$ has the same relative ordering as $\sigma'_j(1) \cdots \sigma'_j(m)$.

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In other words, a d-tuple T_1 contains another d-tuple T_2 if and only if each permutation in T_1 contains the corresponding permutation in T_2 at the same location.

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$\sigma_1 \leq \sigma_2$ in the Weak Bruhat Order.

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If $(\sigma'_1, \dots, \sigma'_d)$ avoids $(\sigma_1, \dots, \sigma_d)$, with $\sigma_1, \dots, \sigma_d \in S_n$, then $[\sigma'_1, \dots, \sigma'_d]$ is a set partition of $[(d+1)n]$ avoiding $[\sigma_1, \dots, \sigma_d]$.

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$$\begin{aligned} Av_n((\sigma_1, \ldots, \sigma_d)) &\leq Av_{(d+1)n}([\sigma_1, \ldots, \sigma_d]) \\ &< c'^n n^{n\left(1-\frac{1}{d}\right)(d+1)} \\ &= c'^n n^{\left(\frac{d^2-1}{d}\right)n} \end{aligned}$$

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Put it all together:

Theorem (G., Pálvölgyi)

Let $\sigma_1, \ldots, \sigma_d \in S_m$, $m > 1$. There exist constants $c' > c > 0$ such that

$$c^n n^{\left(\frac{d^2-1}{d}\right)n} < Av_n((\sigma_1, \ldots, \sigma_d)) < c'^n n^{\left(\frac{d^2-1}{d}\right)n}$$

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Our theorem solves the problem for classes with 1 basis element.
The product of a pattern class of d-tuples with one of d'-tuples gives a pattern class of $d + d'$-tuples.
For d-tuples, can obtain within an exponential factor of $n^{\alpha n}$, for $\alpha = d - \frac{1}{d_1} - \cdots - \frac{1}{d_k}$, $d_i \in \mathbb{Z}^+$, $\sum d_i \leq d$. Can we obtain anything else?

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Can we compute the exponential factors in any of these cases? (Analogous to Stanley-Wilf limits)

- For $Av((12, 12))$, we should have $c^n(n!)^{\frac{3}{2}}$.
- Exponential factor is not currently known!
- $\sqrt{\frac{\pi}{2}}$?

Thank You!