Permutations sortable by deques and by two stacks in parallel.

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Joint work with Tony Guttmann

The University of Melbourne

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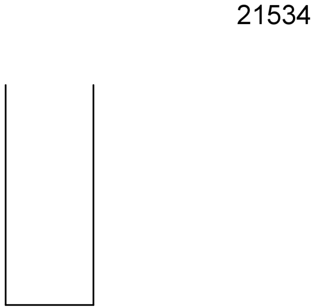

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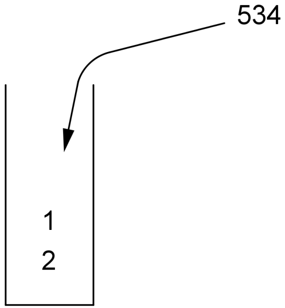

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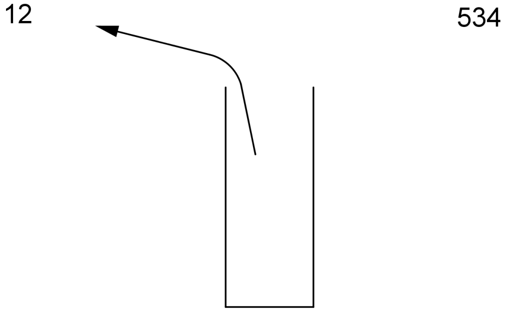

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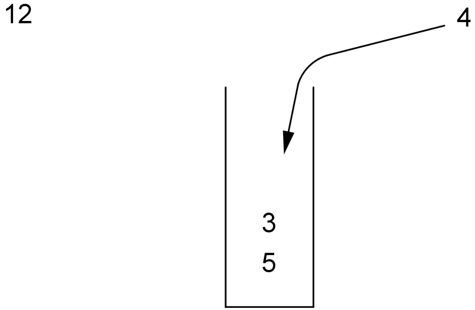

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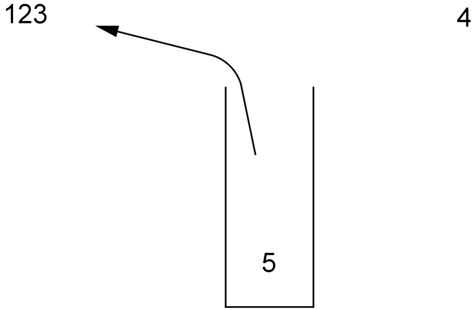

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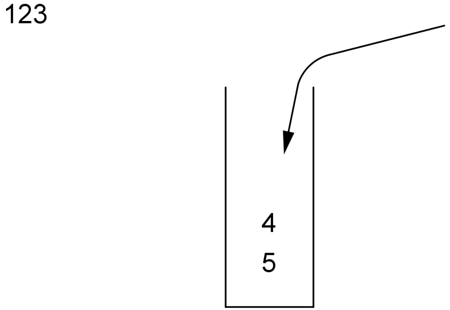

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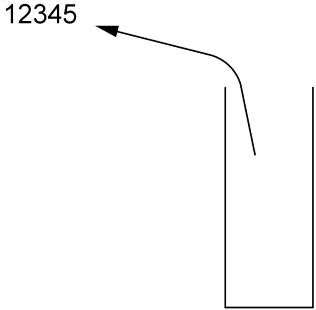

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He also proved that the set of stack-sortable permutations is Av(312).

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Until 2015 none of these two questions had been solved.

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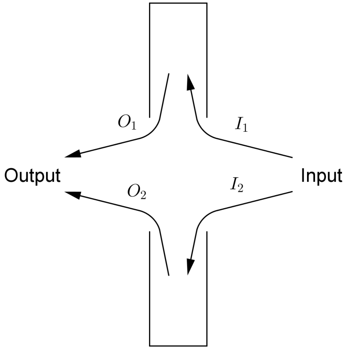
We call these operations I_1, I_2, O_1, O_2 as shown:



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I will give a summary of their solution.

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- Unfortunately, many different operation sequences correspond to the same permutation.
- Call two operation sequences equivalent if they sort the same permutation.
- We want to count the number of equivalence classes of operation sequences.

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This reduces the problem to counting a certain class of quarter plane loops: Those with no *NW* or *ES* corners, in which every sub-(quarter plane loop) begins with *N*.

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They related $P(t)$, the counting function of tsip-sortable permutation, to the generating function $Q(a,u)$ for weighted quarter plane loops, where a counts the number of NW or ES corners, and u counts the halflength.

SOLUTION FOR TWO STACKS IN PARALLEL

They related $P(t)$, the counting function of tsip-sortable permutation, to the generating function $Q(a, u)$ for weighted quarter plane loops, where a counts the number of NW or ES corners, and u counts the halflength.
In particular, they showed that $P(t)$ is characterised by the equation

$$Q\left(\frac{1}{P} - 1, \frac{tP^2}{(2P-1)^2}\right) = 2P - 1$$

Sorting With a Deque

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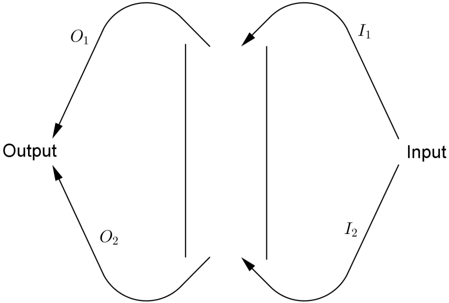
With a deque, There are also four moves I_1, I_2, O_1, O_2, which correspond to input to the top and bottom of the deque and output from the top and bottom of the deque.



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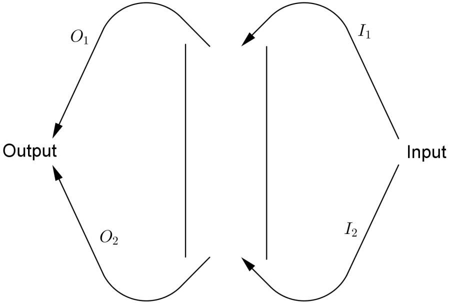

Notice that if we put a wall in the middle of the deque, this becomes equivalent to two stacks in parallel.

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- We associate the operations I_1, I_2, O_1, O_2 with the steps N, E, S, W, respectively.
- The corresponding path of an operation sequence is now restricted to the diagonal half plane $\{(x, y) | x + y \geq 0\}$, and must end on the line $x + y = 0$.

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- The subwords I_1O_2 and I_2O_1 are forbidden.
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- When the deque contains at most 1 element only the moves I_1 and O_1 are allowed, not I_2 or O_2.

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- Any step from a point (x,y) with $x+y \leq 1$ must be N or S.

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- Then we cancelled some functions from the equations...
- Finally, we deduced the remarkable result that the generating functions D for deques and P for two stacks in Parallel are related by the equation

$$D(t) = \frac{t}{2} + 1 + tP - t^2P - \frac{t}{2}\sqrt{1 - 4P + 4P^2 - 8tP^2 + 4t^2P^2 - 4tP}$$

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- Finally $Q(a, u) = W(a, \sqrt{u}, 0, 0)$.

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Our analysis of the coefficients p_n of P suggests that they behave like

$$p_n \sim const \cdot \mu^n n^\gamma,$$

where $\mu \approx 8.281402207$ and $\gamma \approx -2.473$.

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We still can't prove that the two squences $\{p_n\}$ and $\{d_n\}$ share the same exponential growth rate μ.

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it follows pretty easily that the radius of convergence of $D(t)$ is also t_c...**But we still need to prove the conjectures about $Q(a, u)$!!!!**

Quarter plane loops conjecture 1

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Conjecture

the radius of convergence $\rho_Q(a)$ of $Q(a,\cdot)$ is given by

$$\rho_Q(a) = \begin{cases} \dfrac{1}{(2+\sqrt{2+2a})^2}, & \text{if } a \geq -1/2, \\ \dfrac{-a}{2(a-1)^2}, & \text{if } a \in [-1,-1/2]. \end{cases}$$

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The really important bit of the conjecture is at $a \approx -0.148$.

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Conjecture

The series $Q_u(a,u) = \frac{\partial Q}{\partial u}$ is convergent at $u = \rho_Q(a)$ for $a \geq -1/3$.

Open problems

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- Just prove any of the three conjectures about weighted quarter plane loops.
- Find a more direct proof of the identity

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- Four stacks in parallel
- etc
- Two stacks in series
- Two input-restricted deques in parallel
- One input-restricted deque and one stack in parallel
- Any finite system of machines in parallel/series which involves any of the above

THANK YOU

Thank You!