Covering and Upcovering Permutations

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Example 1

What are the permutations $\pi' \in S_3$ that 1324 covers?

- 1324 $\Rightarrow$ 132 is a sub-permutation
- 1324 $\Rightarrow$ 134 $\sim$ 123 is a sub-permutation
- 1324 $\Rightarrow$ 124 $\sim$ 123 is a sub-permutation
- 1324 $\Rightarrow$ 324 $\sim$ 213 is a sub-permutation

132, 123, and 213 are the three-element permutations covered by 1324.

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Example 2

- What are the permutations $\pi' \in S_4$ that 123 "upcovers" or "is contained in?"?
- The answer is $10 = 3^2 + 1$ and $n^2 + 1$ in general, via Lemma 1 in Allison-G-Hawley-Kay (2013).
- The proof relies on arguing that the answer is $\binom{n+1}{n} \cdot (n+1) - 2n = n^2 + 1$.
- So 123 and 321 jointly upcover 20 permutations in S_4.
- No single other permutation upcovers all the remaining 4-permutations, but 231 and 132 can be verified to jointly do the job.
- Thus "$C_{4,3} = 4$", where $C_{n+1,n}$ is the smallest number of n-permutations that must be selected so that each $(n+1)$-permutation contains at least one of the selected permutations.

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Two Natural Questions motivated by a question of Robert Brignall, PP 2011

1. Minimal Order $\kappa_{n,n+k}$

What is the size $\kappa_{n,n+k}$ of a minimal set $A \subseteq S_{n+k}$ such that each $\pi' \in S_n$ is covered by some $\pi \in A$?

That is, what is the minimum number of permutations on $n+k$ elements necessary to cover every permutation on n elements?

Two Natural Questions

2. Minimal Order $C_{n+k,n}$

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That is, what is the minimum number of permutations on n elements necessary to upcover every permutation on $n+k$ elements?

Outline of Work

- The work of Allison et al (2013), referenced earlier, studies the first problem for $k = 1$.
- *In this work* we study $\kappa_{n,n+k}$ for $k \geq 2$.
- *In this work* we also study $C_{n+1,n}$.
- Both sets of results focus on asymptotics as $n \to \infty$ (k is fixed in the first of these two cases).
- The case of $C_{n+k,n}$ for $k \geq 2$ has not yet been addressed.
- We also study threshold behavior for the covering problem, $k \geq 2$ ($k = 1$ done earlier)

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Case: $k = 1$

Results from Allison et al., 2013

For $k = 1$, we have

$$\frac{(n+1)!}{n^2}(1+o(1)) \leq \kappa_{n,n+1} \leq \frac{\log n}{n^2}(n+1)!(1+o(1))$$

The general case for covering n permutations by $n + k$ permutations is similar.

Results for General k

Theorem

Let $k \geq 1$. Then

$$k!\frac{(n+k)!}{n^{2k}}(1+o(1)) \leq \kappa_{n,n+k} \leq k\frac{(n+k)!}{n^{2k}}\log(n)(1+o(1)).$$

The upper bound is proved as follows:

- Randomly select Y permutations in S_{n+k}. These are expected to cover a bunch of n-permutations.
- Find the expected number $Z(Y)$ of *additional* permutations necessary to cover all of S_n
- Minimize the total number of necessary permutations $Y + Z(Y)$ with respect to Y

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- An $n+k$-permutation covers at most $\binom{n+k}{n}$ $\pi' \in S_n$.
- $\kappa_{n,n+k}\binom{n+k}{n} \geq n!$
- for $n \gg k$ we obtain the asymptotic behavior given by the lower bound

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Covering Multiple Times

Suppose we want to cover every element of S_n more than once, i.e., $\lambda \geq 2$ times.

Minimal Order $\kappa_{n,n+k,\lambda}$

What is the order $\kappa_{n,n+k,\lambda}$ of a minimal set $A_\lambda \subseteq S_{n+k}$ such that each $\pi' \in S_n$ is covered λ times by the $\pi \in A$?

Covering λ Times

Results from Allison et al., 2013

For $k = 1$, we have

$$\kappa_{n,n+1,\lambda} \leq \frac{(n+1)!}{n^2}\left(\log n + (\lambda - 1)\log\log n + \frac{\lambda}{(\lambda-1)!}(1+o(1))\right)$$

Theorem

Let $k \geq 2$. Then

$$\kappa_{n,n+k,\lambda} \leq \frac{(n+k)!}{n^{2k}}\left(k\log n + (\lambda - 1)\log[k\log n] + \frac{\lambda}{(\lambda-1)!}(1+o(1))\right)$$

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Successions

Definition

A **succession** is a pair of adjacent elements a and b in a permutation on $[n]$ such that $|a - b| = 1$.

Example:
In the permutation 1243, 1 and 2 form a succession, and 4 and 3 form a succession.

The permutation 3142 does not contain any successions.

Other Results on Successions

Lemma

Each $(n+1)$-permutation π covers $n+1-s_\pi$ n-permutations, where s_π is the numbers of successions in π.

Lemma

Let $n \geq 5$. Let π be a random n-permutation and let j denote the j^{th} element of π. For each $1 \leq j \leq n-1$, let I_j be an indicator function that equals 1 if $(j, j+1)$ is a succession and 0 otherwise. Define $X = \sum_{j=1}^{n-1} I_j$. Then for any constant $k > 0$,

$$\mathbb{P}(X \geq \sqrt{36 n \log n}) \leq \frac{2}{n^4}.$$

Connections to Turán Theory

The classical Mantel-Turán theorem states that the maximum number of edges in a graph on n vertices that does not contain a triangle is $\lfloor n^2/4 \rfloor$, and the extremal graph consists of the complete bipartite graph $K_{a,b}$ with $a = \lfloor n/2 \rfloor$ and $b = \lceil n/2 \rceil$.
There are many extensions to graphs that avoid K_r and to hypergraphs, see, e.g., the LMS survey paper of Keevash (2011). There is another way to view this result by considering the complementary graph: The *minimum* number of edges that must be *absent* so that no triangle is complete *i.e., contains at least one missing edge* is also roughly $n^2/4$, and the extremal graph consists of two edge disjoint copies (if n is even) of $K_{n/2}$. If the graph itself consists of two edge disjoint copies of $K_{n/2}$, then every set of three points contains at least one edge of the graph.

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Turán

It is against this backdrop of Turán type questions that we ask the main question of this section: What is the smallest number of n-permutations that must be selected so that each $(n+1)$-permutation contains at least one of the selected permutations? In this section we find new results on this upcovering problem of all permutations of length $n+1$ with permutations of length n.

Theorem

$C_{3,2} = 2, C_{4,3} = 4$ *and*

$$\frac{(n+1)!}{n^2+1} \leq C_{n+1,n} \leq \frac{2(n+1)!}{n^4} + \frac{2n! \cdot \ln(n+1)}{n+1-7\sqrt{n \log n}}(1+o(1)).$$

In other words,

$$\frac{n!}{n}(1+o(1)) \leq C_{n+1,n} \leq 2\frac{n!}{n} \ln n(1+o(1)).$$

Backwards Covering Theorem: Sketch of Proof

- As before, we have that each n-permutation in the covering set upcovers $n^2 + 1$ $(n+1)$-permutations. Thus, $C_{n+1,n} \cdot (n^2+1) \geq (n+1)!$, which gives the lower bound.
- Partition S_{n+1} into sets $B = \{\pi \in S_{n+1} | s_\pi > 7\sqrt{n \log n}\}$ and $G = S_{n+1} \setminus B$.
- Construct an upcovering set for S_{n+1} by covering B and G separately.
- For each $\pi \in B$, add one n-permutation that backwards covers π to the covering set; there are, by the lemma, at most $2(n+1)!/n^4$ such permutations

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Proof, continued

- To upcover G, proceed as before by picking each n-permutation randomly with some probability p;
- Since each such permutation π in G has no more than $7\sqrt{n \log n}$ successions, there are at least $n + 1 - 7\sqrt{n \log n}$ n-permutations that upcover π
- Calculate expected number of un-upcovered $(n+1)$-permutations and add one n-permutation to Z for each un-upcovered permutation.
- Minimize over p. In other words, use the method of alterations.

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Lemmas

Lemma

For any i there exist at most $O(n^{3k})$ permutations j that can be jointly covered with i by some $\pi \in S_{n+k}$.

Lemma

For any $i, j \in S_n$, there are $O(n^{2k-2})$ permutations in S_{n+k} that cover both.

Thresholds via Janson's Inequality

Theorem

Consider the probability model in which each $\pi \in S_{n+k}$ is independently picked with probability p. Let the resulting random collection of permutations be denoted by $\mathcal{A}$. Then,

$$\lim_{n\to\infty} P(\mathcal{A} \text{ is a cover of } S_n) = 0 \text{ if } p \leq \frac{\log n}{n^{2k-1}}(1+o(1))$$

and

$$\lim_{n\to\infty} P(\mathcal{A} \text{ is a cover of } S_n) = 1 \text{ if } p \geq \frac{\log n}{n^{2k-1}}(1+o^*(1)).$$