Universal permuton limits of substitution-closed permutation classes

Adeline Pierrot

LRI, Univ. Paris-Sud, Univ. Paris-Saclay

Permutation Patterns 2017

ArXiv: 1706.08333

Joint work with Frédérique Bassino, Mathilde Bouvel,

Valentin Féray, Lucas Gerin and Mickaël Maazoun

Main issue

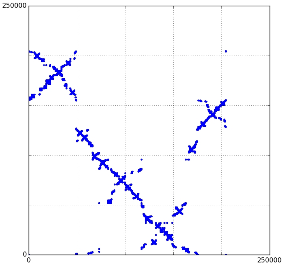
$\mathcal{C}$ a permutation class (i.e $\sigma \in \mathcal{C}$ and $\pi \preccurlyeq \sigma \Rightarrow \pi \in \mathcal{C}$)

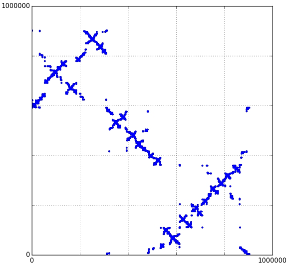
How do typical large permutations in $\mathcal{C}$ look like?

Main issue

$\mathcal{C}$ a permutation class (i.e $\sigma \in \mathcal{C}$ and $\pi \preccurlyeq \sigma \Rightarrow \pi \in \mathcal{C}$)

How do typical large permutations in $\mathcal{C}$ look like?





Separable permutations of size 204523 and 903073,
drawn uniformly at random among those of the same size.

Main issue

$\mathcal{C}$ a permutation class (i.e $\sigma \in \mathcal{C}$ and $\pi \preccurlyeq \sigma \Rightarrow \pi \in \mathcal{C}$)

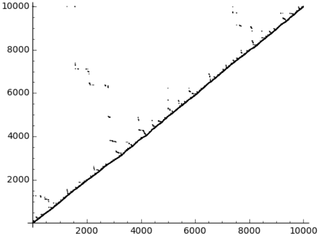
How do typical large permutations in $\mathcal{C}$ look like?

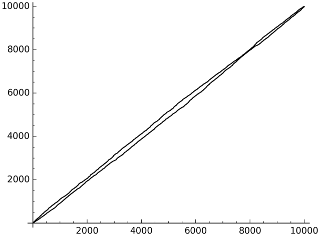
$\boldsymbol{\sigma}_n$ a uniform random permutation in $\mathcal{C}$ of size $n \to \infty$

$\to$ Limit shape of the diagram of $\boldsymbol{\sigma}_n$?

$\to$ Frequency of occurrence of patterns in $\boldsymbol{\sigma}_n$?

Limit shape of permutation diagrams

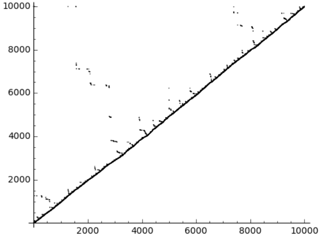


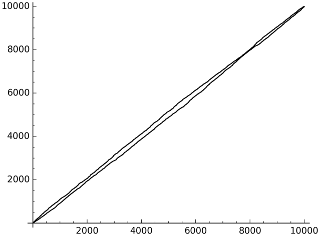


Permutations of size 10 000 in $Av(231)$ and $Av(321)$ [Hoffman Rizzolo Slivken PP2015]

First order limit shape of $Av(231)$:  Same for $Av(321)$.

Limit shape of permutation diagrams



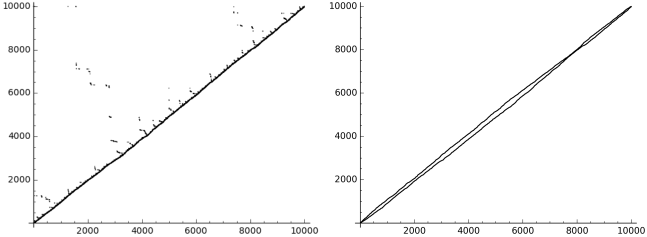


Permutations of size 10 000 in $Av(231)$ and $Av(321)$ [Hoffman Rizzolo Slivken PP2015]

First order limit shape of $Av(231)$:  Same for $Av(321)$.

First order limit shape of $Av(132, 213, 231, 312) = \cup_{n=1}^{\infty}\{12 \dots n, n \dots 21\}$:
 with proba $1/2$ and  with proba $1/2$

Limit shape of permutation diagrams



Permutations of size 10 000 in $Av(231)$ and $Av(321)$ [Hoffman Rizzolo Slivken PP2015]

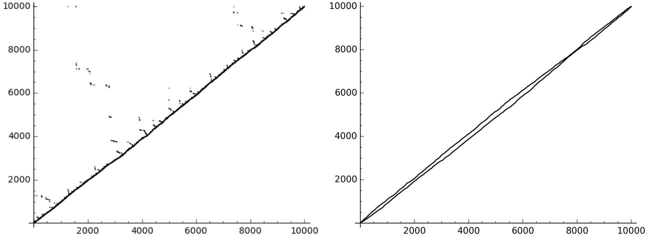
First order limit shape of $Av(231)$:  Same for $Av(321)$.

$\rightarrow$ deterministic limit shape

First order limit shape of $Av(132, 213, 231, 312) = \cup_{n=1}^{\infty} \{12 \dots n, n \dots 21\}$:

 with proba $1/2$ and  with proba $1/2$

Limit shape of permutation diagrams



Permutations of size 10 000 in $Av(231)$ and $Av(321)$ [Hoffman Rizzolo Slivken PP2015]

First order limit shape of $Av(231)$:  Same for $Av(321)$.

$\rightarrow$ deterministic limit shape

First order limit shape of $Av(132, 213, 231, 312) = \cup_{n=1}^{\infty}\{12\dots n, n\dots 21\}$:

 with proba $1/2$ and  with proba $1/2$

$\rightarrow$ non-deterministic limit shape

Pattern densities

Frequency of occurrence of patterns:

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}} \quad \text{for } n = |\sigma|, k = |\pi|$$

$\boldsymbol{\sigma}_n$ a uniform random permutation in $\mathcal{C}$ of size $n \to \infty$

- asymptotics of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)]$?
- limiting distribution for $\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)$?
- joint limiting distribution for $\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)$ for every pattern π?

Pattern densities

Frequency of occurrence of patterns:

$$\widetilde{\mathrm{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}} \quad \text{for } n = |\sigma|, k = |\pi|$$

$\boldsymbol{\sigma}_n$ a uniform random permutation in $\mathcal{C}$ of size $n \to \infty$

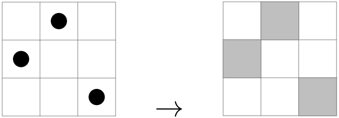
- asymptotics of $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)]$?
- limiting distribution for $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)$?
- joint limiting distribution for $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)$ for every pattern π?

$\to$ linked with limit shapes thanks to permutons

Permutons

A permuton μ is a probability measure on $[0,1]^2$ such that (x,y) drawn under $\mu \Rightarrow x$ (resp. y) is uniform on $[0,1]$.

Permutation $\sigma \Rightarrow$ permuton μ_σ:
normalize the diagram and fill in uniformly cells containing dots



- Permuton approximate permutation diagrams
- $\boldsymbol{\sigma}_n$ random permutation $\Rightarrow \mu_{\boldsymbol{\sigma}_n}$ random permuton

Patterns in permutons

- σ a permutation

$$\widetilde{\text{occ}}(\pi,\sigma) = \frac{\sharp \text{ occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}} = \mathbb{P}\left(\text{pat}_{\boldsymbol{I}}(\sigma) = \pi\right)$$

with $\boldsymbol{I}$ a uniform random subset of $[n]$ with k elements

- μ a permuton

$\widetilde{\text{occ}}(\pi,\mu)$ = probability that k points drawn from μ are isomorphic to the diagram of π

Random permutons convergence

Theorem:

$(\boldsymbol{\sigma}_n)$ random permutations of size n. The following are equivalent:

- $\mu_{\boldsymbol{\sigma}_n}$ converges in distribution to some random permuton $\boldsymbol{\mu}$
- $\big(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\big)_{\pi \in \mathfrak{S}}$ converges in distribution to some random infinite vector $(\boldsymbol{\Lambda}_\pi)_{\pi \in \mathfrak{S}}$.
- $\forall \pi \in \mathfrak{S}$, $\exists\, \Delta_\pi \geq 0$ s.t. $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n\to\infty} \Delta_\pi$

Then $(\boldsymbol{\Lambda}_\pi)_\pi \stackrel{d}{=} (\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu}))_\pi$ and $\forall \pi \in \mathfrak{S}$, $\Delta_\pi = \mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu})]$

Random permutons convergence

Theorem:

$(\boldsymbol{\sigma}_n)$ random permutations of size n. The following are equivalent:

- $\mu_{\boldsymbol{\sigma}_n}$ converges in distribution to some random permuton $\boldsymbol{\mu}$
- $\left(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right)_{\pi \in \mathfrak{S}}$ converges in distribution to some random infinite vector $(\boldsymbol{\Lambda}_\pi)_{\pi \in \mathfrak{S}}$.
- $\forall \pi \in \mathfrak{S}$, $\exists\, \Delta_\pi \geq 0$ s.t. $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n \to \infty} \Delta_\pi$

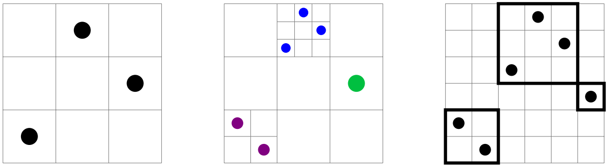
Then $(\boldsymbol{\Lambda}_\pi)_\pi \stackrel{d}{=} (\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu}))_\pi$ and $\forall \pi \in \mathfrak{S}$, $\Delta_\pi = \mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu})]$

Goal:

Find the permuton limit of $(\boldsymbol{\sigma}_n)$ uniform random permutations in a substitution-closed class.

Substitution

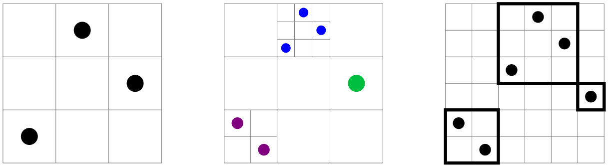
Substitution $\sigma[\pi^{(1)}, \ldots, \pi^{(n)}]$: Replace each point σ_i by a block $\pi^{(i)}$



Example : $1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Substitution

Substitution $\sigma[\pi^{(1)}, \ldots, \pi^{(n)}]$: Replace each point σ_i by a block $\pi^{(i)}$

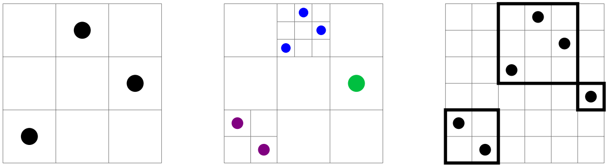


Example : $1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Remark : $\sigma[\pi^{(1)}, \ldots, \pi^{(n)}] \in \mathcal{C} \Rightarrow \sigma, \pi^{(1)}, \ldots, \pi^{(n)} \in \mathcal{C}$

Substitution

Substitution $\sigma[\pi^{(1)}, \dots, \pi^{(n)}]$: Replace each point σ_i by a block $\pi^{(i)}$



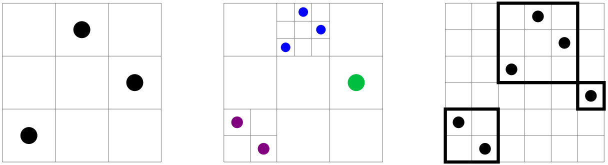
Example : $1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Remark : $\sigma[\pi^{(1)}, \dots, \pi^{(n)}] \in \mathcal{C} \;\Rightarrow\; \sigma, \pi^{(1)}, \dots, \pi^{(n)} \in \mathcal{C}$

Substitution-closed class:
$\sigma, \pi^{(1)}, \dots, \pi^{(n)} \in \mathcal{C} \;\Rightarrow\; \sigma[\pi^{(1)}, \dots, \pi^{(n)}] \in \mathcal{C}$.

Substitution

Substitution $\sigma[\pi^{(1)}, \ldots, \pi^{(n)}]$: Replace each point σ_i by a block $\pi^{(i)}$



Example : $1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Remark : $\sigma[\pi^{(1)}, \ldots, \pi^{(n)}] \in \mathcal{C} \;\Rightarrow\; \sigma, \pi^{(1)}, \ldots, \pi^{(n)} \in \mathcal{C}$

Substitution-closed class:
$\sigma, \pi^{(1)}, \ldots, \pi^{(n)} \in \mathcal{C} \;\Rightarrow\; \sigma[\pi^{(1)}, \ldots, \pi^{(n)}] \in \mathcal{C}$.

Simple permutation = indecomposable:
α simple $\Leftrightarrow$ cannot be written as $\sigma[\pi^1, \ldots, \pi^n]$ with $1 < |\sigma| < |\alpha|$

Permutations $\Leftrightarrow$ trees

Every permutation σ of size $n \geq 2$ can be uniquely decomposed as either:

- $\alpha[\pi^{(1)}, \ldots, \pi^{(d)}]$ where α is simple of size $d \geq 4$
- $\oplus[\pi^{(1)}, \ldots, \pi^{(d)}]$ where $d \geq 2$ and $\pi^{(1)}, \ldots, \pi^{(d)}$ are $\oplus$-indecomposable
- $\ominus[\pi^{(1)}, \ldots, \pi^{(d)}]$ where $d \geq 2$ and $\pi^{(1)}, \ldots, \pi^{(d)}$ are $\ominus$-indecomposable

Permutations $\Leftrightarrow$ trees

Every permutation σ of size $n \geq 2$ can be uniquely decomposed as either:

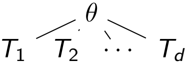
- $\alpha[\pi^{(1)}, \ldots, \pi^{(d)}]$ where α is simple of size $d \geq 4$
- $\oplus[\pi^{(1)}, \ldots, \pi^{(d)}]$ where $d \geq 2$ and $\pi^{(1)}, \ldots, \pi^{(d)}$ are $\oplus$-indecomposable
- $\ominus[\pi^{(1)}, \ldots, \pi^{(d)}]$ where $d \geq 2$ and $\pi^{(1)}, \ldots, \pi^{(d)}$ are $\ominus$-indecomposable

Canonical tree: rooted planar tree whose internal nodes have labels s.t.

- Internal nodes are labeled by $\oplus$, $\ominus$, or by a simple permutation.
- A node labeled by α has degree $|\alpha|$, nodes labeled by $\oplus$ and $\ominus$ have degree at least 2.
- A child of a node labeled by $\oplus$ (resp.$\ominus$) cannot be labeled by $\oplus$ (resp.$\ominus$).

Bijection: permutation $\sigma \leftrightarrow$ canonical tree T_σ:

$\sigma = \theta[\pi^{(1)}, \ldots, \pi^{(d)}] \;\Leftrightarrow\; T_\sigma =$



where $T_i = T_{\pi^{(i)}}$

Convenient description of substitution-closed classes

$\mathcal{S}$ a (finite or infinite) set of simple permutations

$\langle \mathcal{S} \rangle = \{\sigma \mid T_\sigma$ has only nodes $\oplus, \ominus$ and $\alpha \in \mathcal{S}\}$

$\mathcal{S}$ downward-closed $= \forall \sigma \in \mathcal{S}, \forall$ simple $\pi \preccurlyeq \sigma$, then $\pi \in \mathcal{S}$

Convenient description of substitution-closed classes

$\mathcal{S}$ a (finite or infinite) set of simple permutations

$\langle \mathcal{S} \rangle = \{\sigma \mid T_\sigma$ has only nodes $\oplus, \ominus$ and $\alpha \in \mathcal{S}\}$

$\mathcal{S}$ downward-closed $= \forall \sigma \in \mathcal{S}, \forall$ simple $\pi \preccurlyeq \sigma$, then $\pi \in \mathcal{S}$

$\mathcal{C}$ substitution-closed class $\Leftrightarrow \mathcal{C} = \langle \mathcal{S} \rangle$ for some downward-closed $\mathcal{S}$

Ex: separable permutations $= \langle \varnothing \rangle$

$\mathcal{S}$ not downward-closed $\Rightarrow \langle \mathcal{S} \rangle$ is not a permutation class,
but results of this talk still true for this kind of sets.

Main result: Standard case

$\mathcal{S}$ a (finite or infinite) set of simple permutations

$$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|} \quad , \quad R_S \in [0, +\infty] \text{ its radius of convergence}$$

$\boldsymbol{\sigma}_n$ a uniform permutation in $\langle \mathcal{S} \rangle_n$ $\forall n \geq 1$

Under some condition $(H1)$, $(\mu_{\boldsymbol{\sigma}_n})_n \xrightarrow{d} \boldsymbol{\mu}^{(p)}$, the biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in the elements of $\mathcal{S}$.

Condition $(H1)$: $R_S > 0 \quad \text{and} \quad \lim_{\substack{r \to R_S \\ r < R_S}} S'(r) > \frac{2}{(1+R_S)^2} - 1$

Why "Standard case"?

$\rightarrow$ covers many natural cases:

- $R_S > \sqrt{2} - 1$, in particular $\mathcal{S}$ finite or s_n grows subexponentially (bounded or polynomial)
- S' divergent at R_S, in particular, S rational generating function, or S with a square root singularity at R_S.

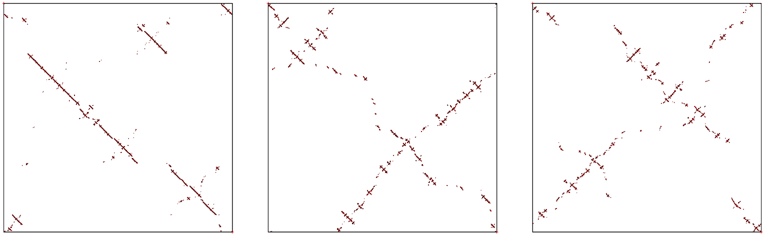
Why "Standard case"?

→ covers many natural cases:

- $R_S > \sqrt{2} - 1$, in particular $\mathcal{S}$ finite or s_n grows subexponentially (bounded or polynomial)
- S' divergent at R_S, in particular, S rational generating function, or S with a square root singularity at R_S.

→ all sets $\mathcal{S}$ studied in the literature enters the standard case!

The biased Brownian separable permuton



Simulations of $\boldsymbol{\mu}^{(p)}$ for $p = 0.2$, $p = 0.45$ and $p = 0.5$

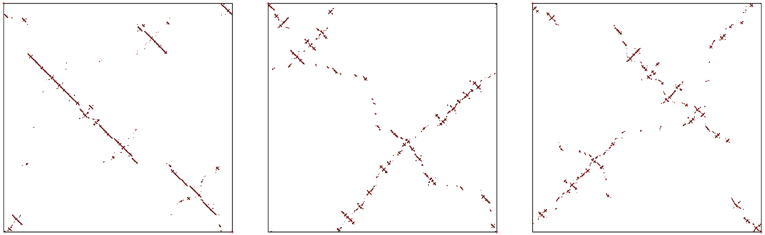
$\boldsymbol{\mu}^{(p)}$ characterized by

$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu}^{(p)})] = \frac{N_\pi}{\mathrm{Cat}_{k-1}}\ p^{r_+(\pi)}\,(1-p)^{r_-(\pi)}$ $\forall k \geq 2$ and $\forall \pi \in \mathfrak{S}_k$

with $N_\pi = \sharp$ separation trees of π ($= 0$ if π non-separable!) and

$r_+(\pi)$ (resp. $r_-(\pi)$) $= \sharp$ nodes labeled $\oplus$ (resp. $\ominus$) in such a tree.

The biased Brownian separable permuton



Simulations of $\boldsymbol{\mu}^{(p)}$ for $p = 0.2$, $p = 0.45$ and $p = 0.5$

$\boldsymbol{\mu}^{(p)}$ characterized by

$\mathbb{E}[\widetilde{\text{occ}}(\pi, \boldsymbol{\mu}^{(p)})] = \frac{N_\pi}{\text{Cat}_{k-1}} \; p^{r_+(\pi)} (1-p)^{r_-(\pi)}$ $\forall k \geq 2$ and $\forall \pi \in \mathfrak{S}_k$
with $N_\pi = \sharp$ separation trees of π ($= 0$ if π non-separable!) and
$r_+(\pi)$ (resp. $r_-(\pi)$) $= \sharp$ nodes labeled $\oplus$ (resp. $\ominus$) in such a tree.

$\boldsymbol{\mu}^{(p)}$ can be directly build from the signed Brownian excursion

Degenerate case

Case $S'(R_S) < 2/(1+R_S)^2 - 1$, with a condition (CS)

If uniform simple permutations in $\mathcal{S}$ have a permuton limit then the limit of uniform permutations in $\langle \mathcal{S} \rangle$ is the same.

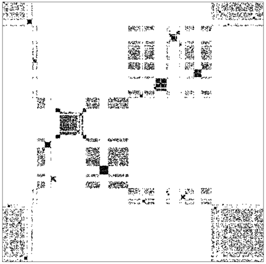
$\rightarrow$ Degenerate case

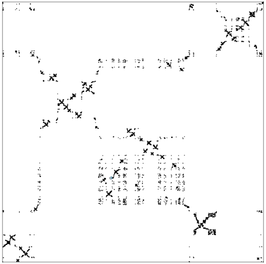
Critical case

Case $S'(R_S) = 2/(1+R_S)^2 - 1$, with condition (CS)

According to the behavior of S near R_S, the permuton limit of $\boldsymbol{\sigma}_n$ is

- either a biased Brownian separable permuton
- or a stable permuton, defined using the random *stable tree*





Simulations of a 1.1-stable and 1.5-stable permuton

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.
- In the degenerate case, the limit is the same as the one of simple permutations of $\mathcal{C}$.

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.
- In the degenerate case, the limit is the same as the one of simple permutations of $\mathcal{C}$.
- In the critical case, the limit is either biased Brownian separable permuton, or a stable permuton.

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.
- In the degenerate case, the limit is the same as the one of simple permutations of $\mathcal{C}$.
- In the critical case, the limit is either biased Brownian separable permuton, or a stable permuton.
- Results also true for sets of permutations that are not permutation classes, but can be described with labels of canonical trees.

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.
- In the degenerate case, the limit is the same as the one of simple permutations of $\mathcal{C}$.
- In the critical case, the limit is either biased Brownian separable permuton, or a stable permuton.
- Results also true for sets of permutations that are not permutation classes, but can be described with labels of canonical trees.
- Are there substitution-closed classes that do enters the standard case?

Conclusion and Perspectives

- We give the permutons limits of large families of substitution-closed classes $\mathcal{C}$.
- In the standard case, the limit is a biased Brownian separable permuton whose parameter p only depends on the quantity of occurrences of 12 and 21 in simple permutations of $\mathcal{C}$.
- In the degenerate case, the limit is the same as the one of simple permutations of $\mathcal{C}$.
- In the critical case, the limit is either biased Brownian separable permuton, or a stable permuton.
- Results also true for sets of permutations that are not permutation classes, but can be described with labels of canonical trees.
- Are there substitution-closed classes that do enters the standard case?

Thank you!