The Thesis That Wasn't

Jennifer Galovich
College of St. Benedict/St. John's University

Open Problems
PP 2017

A permutation statistic

Let w be in S_n

Compute $\overrightarrow{inv}$ w, the left to right inversion table of w.

Weight the entries of $\overrightarrow{inv}$ w by position, and call that new index G(w):

w =7413652

$\overrightarrow{inv}$ w =6301210

G(w) = $\overrightarrow{inv}$ w • (1,2,…,n)

= (6,3,0,1,2,1,0) •(1,2,3,4,5,6,7)

= 6+6+0+4+10+6+0 =32

GF? Look at n=5 case:

$$(1+q+q^2+q^3+q^4)(1+q^2+q^4+q^6)(1+q^3+q^6)(1+q^4)(1)$$

The first number in the inversion vector can be 0, 1, 2, 3 or 4 -- weighted by 1

The second number in the inversion vector can be 0, 1, 2 or 3 --weighted by 2

The third number in the inversion vector can be 0, 1 or 2 --weighted by 3,

The GF of the G statistic is

$$[n]_q[n-1]_{q^2}\cdots[2]_{q^{n-1}}[1]_q$$

$$G(q) = [n]_q [n-1]_{q^2} \cdots [2]_{q^{n-1}} [1]_q$$

- G is symmetric
- Conjecture: G is unimodal

  (Verified for $n \leq 200$)

Idea #1

- If w is in S_n , prepend *n+1* . Count the number of occurrences of 321, 231 and 132 in the extended word. This is equivalent to counting the number of inversions weighted by position.

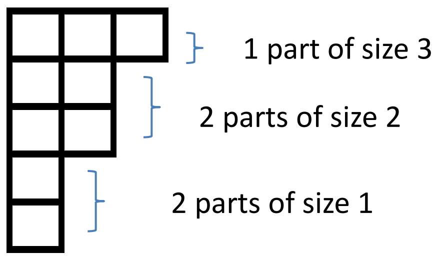
Try to show unimodality by finding an injection.

Idea #2

Represent the weighted inversion vectors as certain Ferrers diagrams, and make a poset.

$(b_1,b_2,\ldots,b_{n-1},0) \longrightarrow b_1\ 1\text{s}, b_2\ 2\text{s},\ldots,b_{n-1}\ n\text{-}1\text{s}$

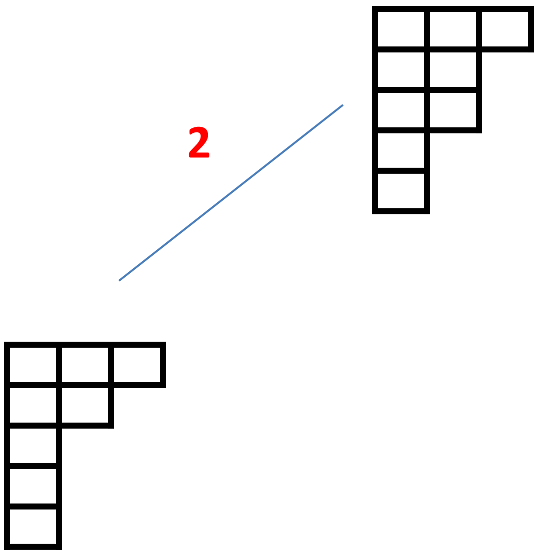
$w=3421 \longrightarrow \overrightarrow{inv}w = (2,2,1,0) \longrightarrow$



Note that the total number of boxes is exactly the value of G for that inversion vector:

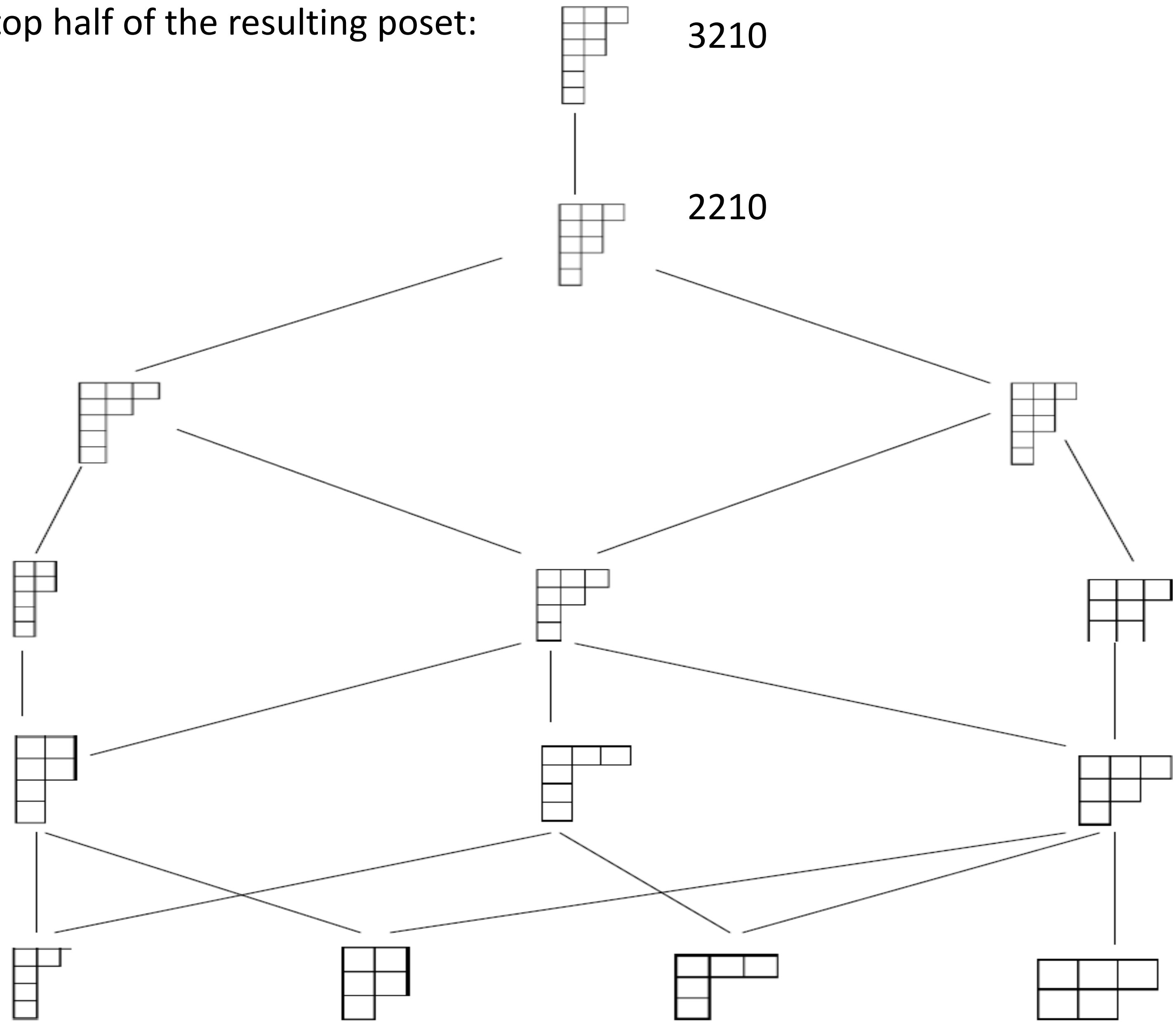
$$G(3421)= (2,2,1,0)\cdot (1,2,3,4) = 9$$

Now partially order the diagrams by inclusion.



The bottom diagram fits inside the upper one, which has one additional box. We can keep track of how that box was added. In this case, we traded in a part of size 1 for a part of size 2. So we will give the connecting edge the label **2**.

Here's the top half of the resulting poset:



ETC.

And here's the whole picture:
Note that the ranks are the coefficie
G.

N = 4

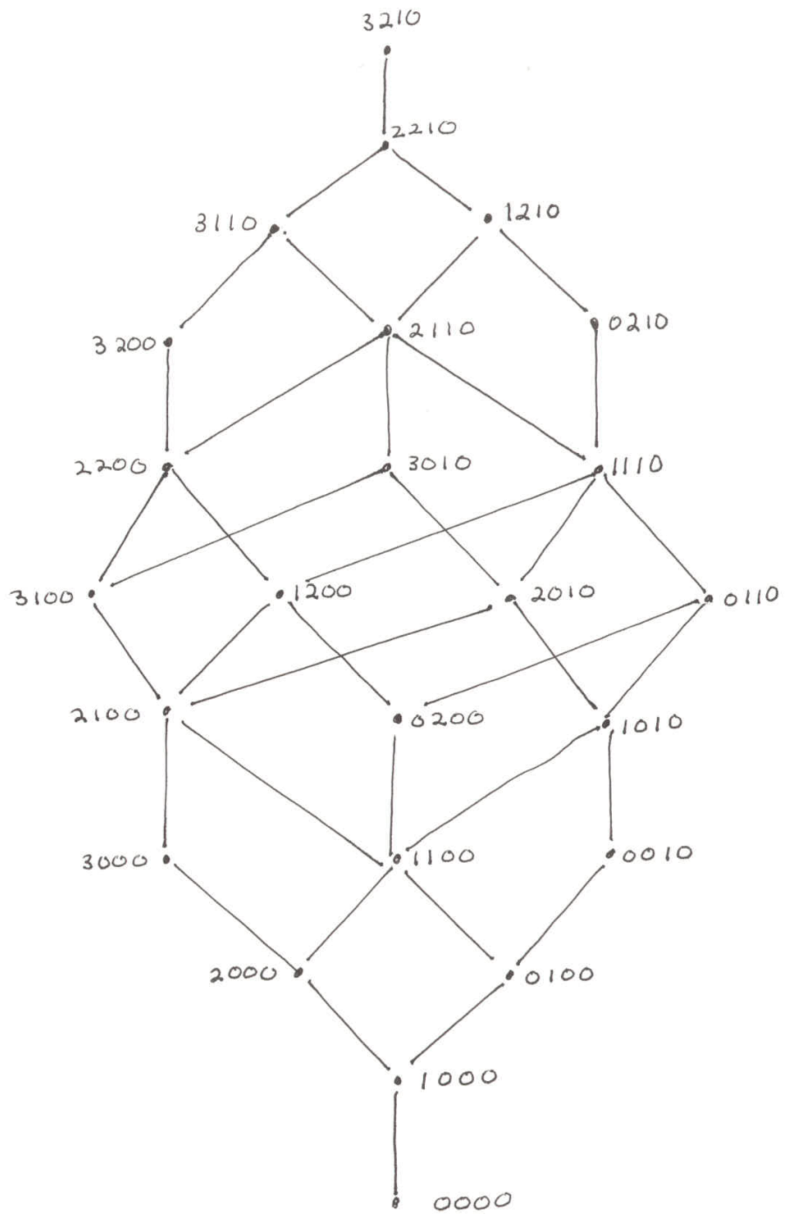
The ranks are

1,1,2,3,3,4,3,3,2,1,1

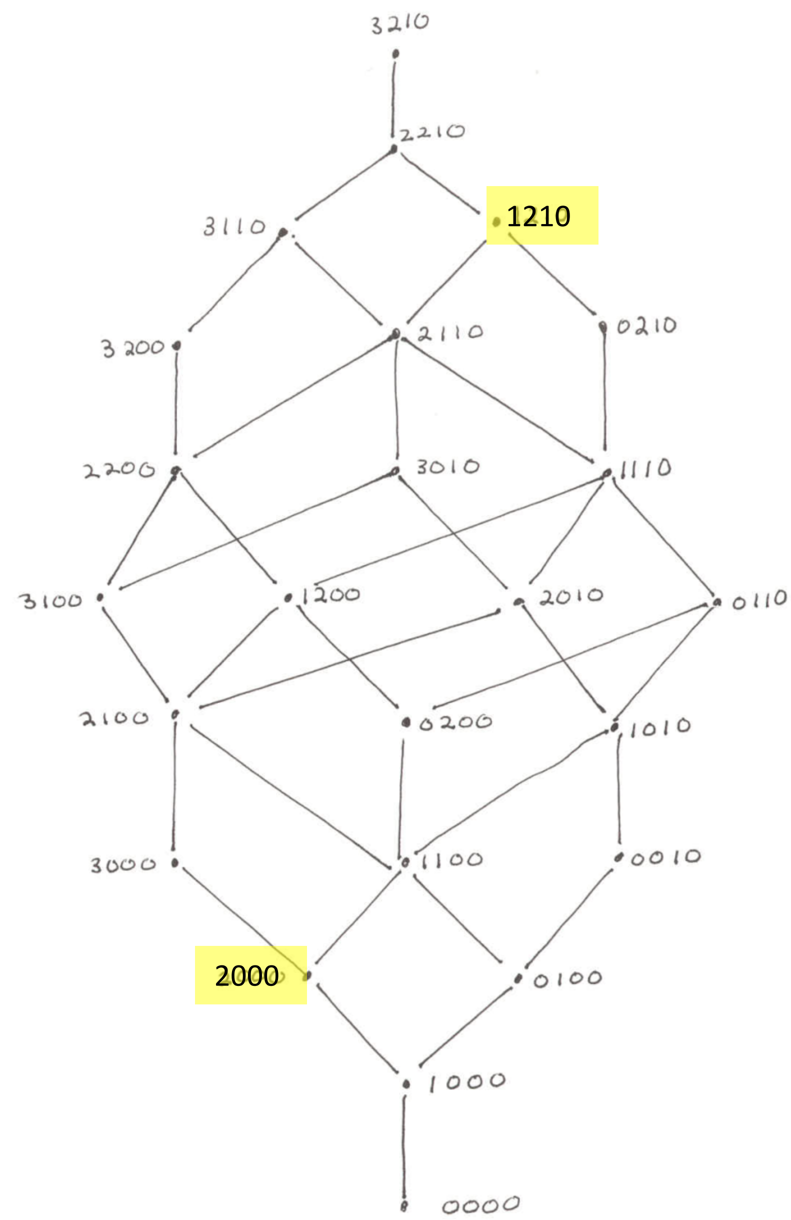
The GF is

$= 1\,q^{0} + 1q^{1} + 2q^{2} + 3q^{3}$
$+ 3q^{4} + 4q^{5} + 3q^{6} + 3q^{7} + 2q^{8}$
$+ 1q^{9} + 1\,q^{10}$

$= [4]_q [3]_{q^2} [2]_{q^3}$



Symmetric? Yes – take complement



Unimodal?

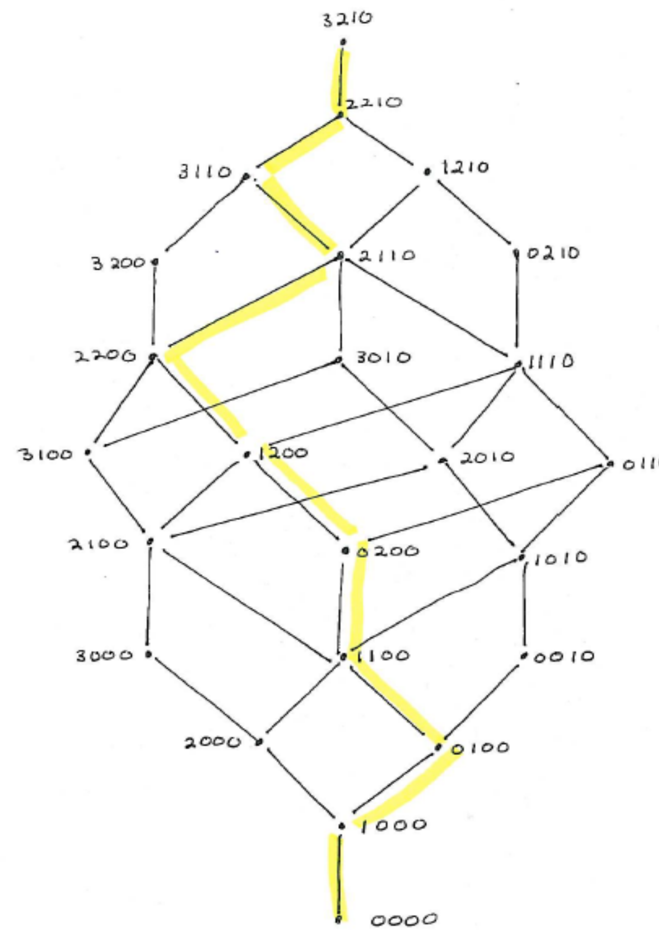
Again, look for an injection.

Idea #3

- Inflict the problem on a talented student.

What else could be learned? Look at maximal chains

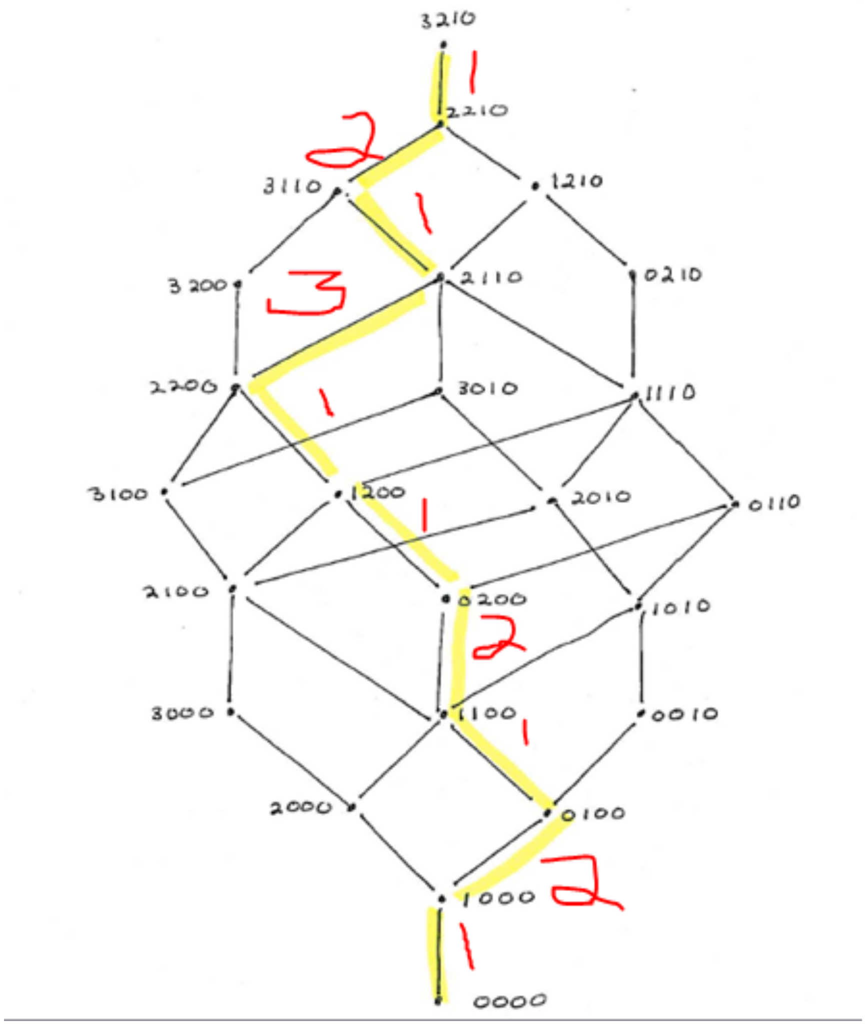
A maximal chain runs from the top to the bottom, hitting every level in between.



Given a maximal chain,

label each edge by the size of the new part, to get a description of the chain:

1213112121



Fact

The maximal chains in this poset are lattice words

(aka Yamanouchi words aka ballot sequences)

Read left to right, at any point, the number of 1s beats or ties the number of 2s, which beats or ties the number of 3s, etc.

Our example: 1213112121

But not 1 2 2 1 1 2 3 1 1 1

BAD!!

Why there is hope:

M(n)

- elements of *M(n)* are subsets of *{1,2,…,n}*, written as *n*-tuples of 0s and 1s

{2,3} ⟷ (0,1,1,0)

- ranked by set sums

(0,1,1,0) is rank 5

- Maximal chains are lattice words (ballot sequences)
- Known to be unimodal!!

My poset P(n)

- Elements are inversion vectors
- ranked by multiset sums

(2,2,1,0) is rank 7 = (2)(1)+(2)(2)+(1)(3)+(0)(4)

- Maximal chains are lattice words (ballot sequences)
- Unimodality???

Have at it!!

(And send your proofs to jgalovich@csbsju.edu)