Coincidence classification of length 3 mesh patterns

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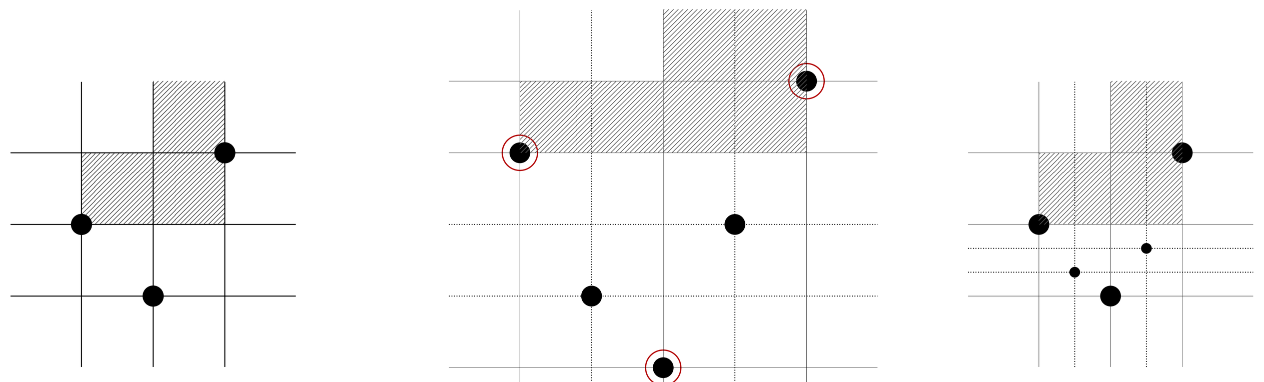
Abstract

Two mesh patterns p and q are *coincident*, denoted $p \asymp q$, if $\mathrm{Av}(p) = \mathrm{Av}(q)$. We give a full coincidence classification of mesh patterns of length at most three. Every coincidence can be proved with an algorithm we introduce, except for one case.

Mesh Patterns



The graph of the mesh pattern $(213, \{(1,2),(2,2),(2,3)\})$; an occurrence of this pattern within the permutation 42135; and a graph focusing on the occurrence, showing the locations of the remaining points in the permutation.

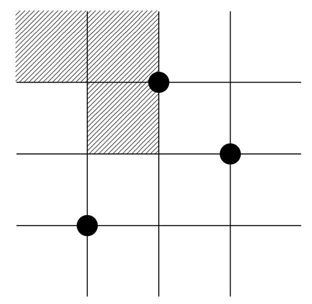
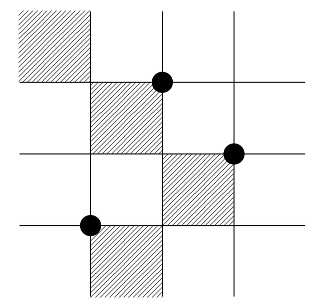
The force

Given a mesh pattern $p = (\tau, R)$, with $\tau \in S_k$, we define a *force* on it as a tuple $((\tau_{i_1}, D_1), (\tau_{i_2}, D_2), \dots, (\tau_{i_\ell}, D_\ell))$ where $0 \leqslant \ell \leqslant k$, the indices $1 \leqslant i_j \leqslant k$ are distinct, and $D_j \in \{N, E, S, W\}$ represents the direction we are forcing the point τ_{i_j} in.

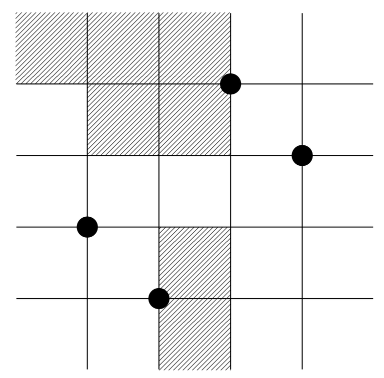
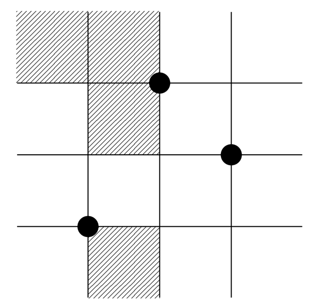
This definition makes it easier to talk about an occurrence of, say, 132 having 3 as far to the West as possible, then out of all those occurrences the 2 as far South as possible. This would be recorded as an occurrence of 132 maximizing the force $((3, W), (2, S))$.

The shading algorithm

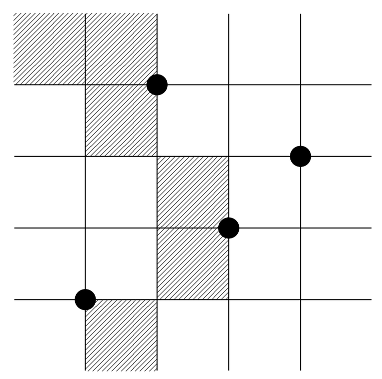
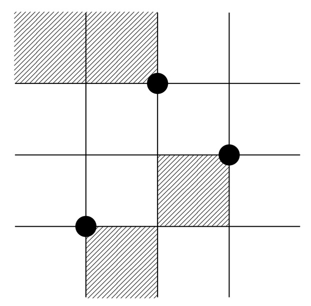
We want to show that an occurrence of the pattern $p = (\tau, R)$ maximizing the force $F = ((1, E))$ implies an occurrence of p'

$p =$  $p' =$ 

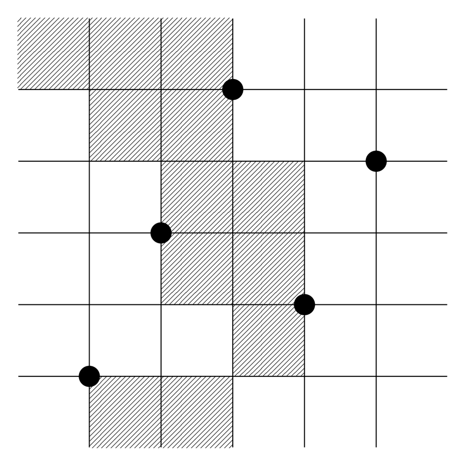
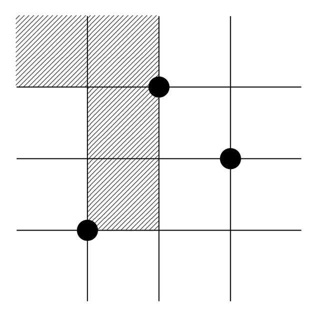
We need to shade $S = \{(1,0),(2,1)\}$. We first consider the rightmost point in $(1,0)$.

$w_1 = p^{\overrightarrow{(1,0)}} =$  $q_1 =$ 

We take the subsequence at indices 234. This is a stronger occurrence of p, a contradiction. We now consider

$w_2 = (\tau, R \cup (1,0))^{\overleftarrow{(2,1)}} =$  $q_2 =$ 

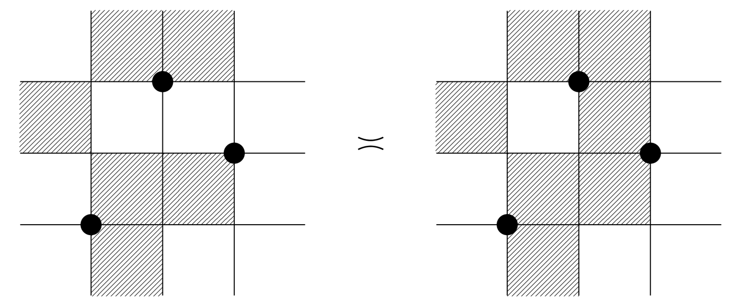
We take the subsequence at indices 123. This gives q_2 above. We need to consider the box $(1,2)$, which corresponds to $(1,2)$ in w_2. We insert the rightmost point in $(1,2)$:

$w_3 = w_2^{\overrightarrow{(1,2)}} =$  $q_3 =$ 

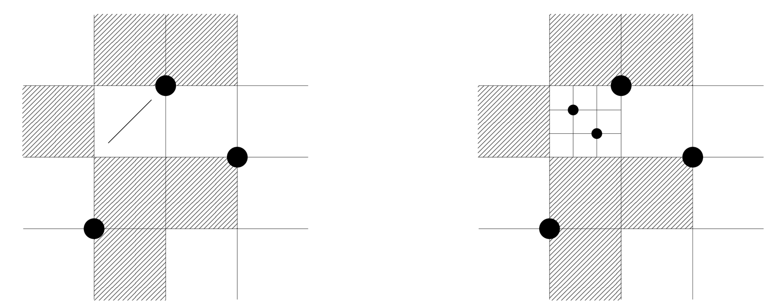
In here we take the subsequence at indices 235. This is a stronger occurrence of p. Hence, if the region S is non-empty we find a stronger occurrence of p (with respect to the force), a contradiction.

The exceptional case

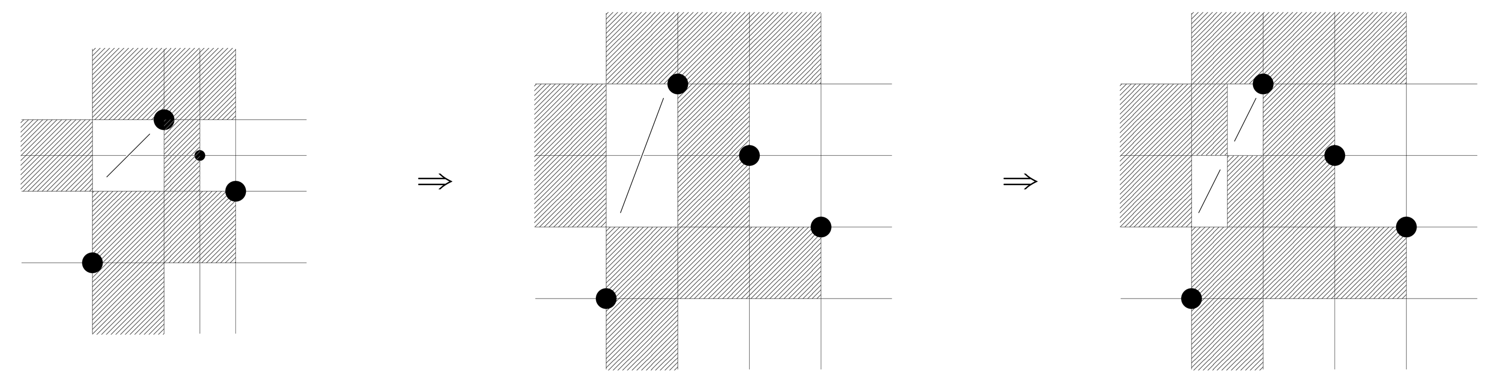
We want to show that



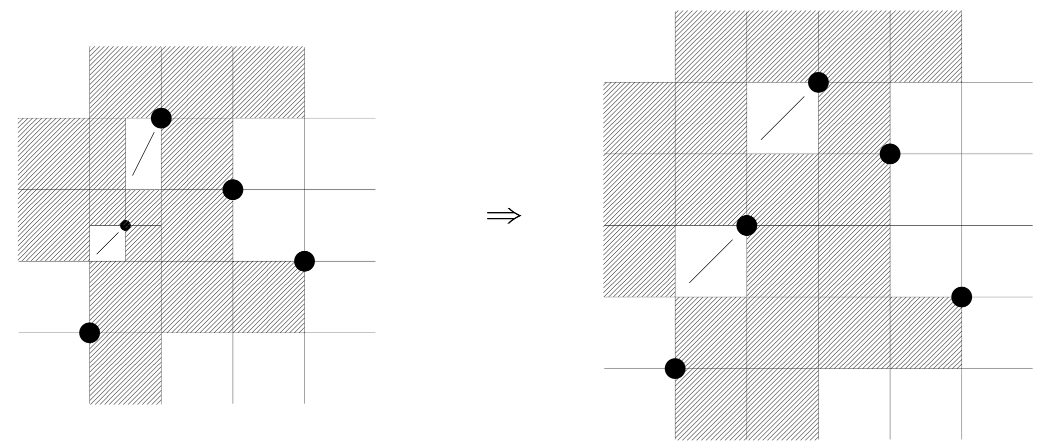
We use the force $((2, E))$. We consider two cases, either the box $(1,2)$ contains an increasing sequence or an inversion, a decreasing pair. We start with the first case.



If the box $(2,2)$ is empty we are done, so assume we can pick the leftmost point.



If the box $(1,2)$ is empty we are done, so assume we can pick the highest point, which is also the rightmost point. The points at indices 234 form a stronger occurrence of the pattern.



The second case relies on showing that the following two patterns are coincident and leads to a similar contradiction.