Pattern Avoiding Generalized Alternating Permutations

Kristina Garrett, St. Olaf College and Kendra Killpatrick, Pepperdine University

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Introduction

Let $S_n(\sigma)$ denote the set of permutations in S_n that avoid the pattern σ. In one of the best known results in the theory of pattern avoiding permutations, Knuth proved in 1968 that the number of permutations in S_n avoiding the pattern 123 is given by the nth Catalan number, C_n.

Theorem 1 (Knuth,1968).

$$|S_n(\sigma)| = C_n \text{ for any pattern } \sigma \text{ of length 3.}$$

The nth Catalan number C_n also counts the number of standard Young tableaux of shape $< 2^n >$, which is denoted by $f^{<2^n>}$. Thus for any pattern σ of length 3,

$$|S_n(\sigma)| = f^{<2^n>}.$$

In 2011, Lewis [2] extended this idea to alternating permutations, i.e. permutations

$$\pi = \pi_1\pi_2\cdots\pi_n \text{ such that } \pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots.$$

Lewis proved that the number of alternating permutations of length $2n$ avoiding the pattern 1234 is counted by the number of standard Young tableaux of shape $< 3^n >$.

Theorem 2 (Lewis, 2011).

$$|A_{2n}(1234) = f^{<3^n>}.$$

Lewis generalized further to the set $L_{n,k}$ of permutations

$$\pi = \pi_{11}\pi_{12}\pi_{13}\cdots\pi_{1k}\pi_{21}\pi_{22}\cdots\pi_{2k}\cdots\pi_{n1}\pi_{n2}\cdots\pi_{nk}$$

of length nk such that $\pi_{i1} < \pi_{i2} < \cdots < \pi_{ik}$ for $1 \leq i \leq n$ by proving that the number of permutations in $L_{n,k}(123\cdots k(k+1)(k+2))$ is counted by the number of standard Young tableaux of shape $< (k+1)^n >$.

Theorem 3 (Lewis, 2011).

$$|L_{n,k}(123\cdots k(k+1)(k+2))| = f^{<(k+1)^n>}.$$

Background

We consider the set $L_{n,k,I}(123\cdots(k+1)(k+2)$ of pattern avoiding permutations studied by Mei and Wang in 2017 [1]. For ease of notation, let $\mathcal{L}(n,k,I)$ mean $L_{n,k,I}(123\cdots(k+1)(k+2)$.

Definition 1. Let n, k be positive integers and let I be an index set $I \subseteq [n]$. Then $\mathcal{L}(n,k,I)$ is the set of permutations $\sigma \in S_{nk+|I|}$ such that

$$\sigma = \sigma_{11}\sigma_{12}\cdots\sigma_{1j_1}\sigma_{21}\cdots\sigma_{2j_2}\cdots\sigma_{n1}\cdots\sigma_{nj_n}$$

where

(C1) $j_i = k+1$ if $i \in I$ and $j_i = k$ otherwise

(C2) $\sigma_{i1} < \sigma_{i2} < \cdots < \sigma_{ij_i}$ for all $1 \leq i \leq n$

(C3) σ avoids the pattern $123\cdots(k+1)(k+2)$.

Example 1. Consider the set $\mathcal{L}(5,2,\{1,3\})$ with $n=5$, $k=2$ and $I=\{1,3\}$. Then $nk+|I| = 12$ so by Definition 1, the set contains (1234)-avoiding permutations in S_{12} that can be split into 5 **blocks**, with blocks one and three of length three and the remaining three blocks of length two. By definition, the elements in each block are increasing from left to right. We will represent such a permutation by indicating the blocks using underbraces. One such element $\sigma \in \mathcal{L}(5,2,\{1,3\})$ is given by:

$$\sigma = \underbrace{7\ 10\ 12}\ \underbrace{6\ 9}\ \underbrace{1\ 5\ 11}\ \underbrace{3\ 8}\ \underbrace{2\ 4}.$$

Mei and Wang [1] extend Lewis's bijection to a bijection between $\mathcal{L}(n,k,I)$ and the set of standard Young tableaux of shape $< (k+1)^n >$ for any set $I \subseteq [n]$.

Theorem 4 (Mei and Wang, 2017). *For any index set $I \subseteq [n]$,*

$$\mathcal{L}(n,k,I) = f^{<(k+1)^n>}.$$

Corollary 5. *For any two index sets I and I' that are subsets of $[n]$,*

$$\mathcal{L}(n,k,I) = f^{<(k+1)^n>} = \mathcal{L}(n,k,I').$$

I.e., $|\mathcal{L}(n,k,I)|$ is independent of the choice of I. This raises the question of finding a bijection between $|\mathcal{L}(n,k,I)|$ and $|\mathcal{L}(n,k,I')|$ for any two index sets I and I' in $[n]$. Mei and Wang's [1] bijection (Theorem 2.3) between $|\mathcal{L}(n,k,I)|$ and the set of standard Young tableaux of shape $< (k+1)^n >$ utilizes the RSK correspondence. Composing this bijection and its inverse can give a mapping between $|\mathcal{L}(n,k,I)|$ and $|\mathcal{L}(n,k,I')|$, however this map is not intuitive and is a bit technical in the details.

Open Question: Is there a direct bijection between $|\mathcal{L}(n,k,I)|$ and $|\mathcal{L}(n,k,I')|$ that doesn't make use of the standard Young tableaux?

Yes!

Results

In [1], the authors give the following bijection between $\mathcal{L}(n,1,\{1\})$ and $\mathcal{L}(n,1,\emptyset)$.

Let $\sigma = \sigma_{11}\sigma_{12}\sigma_{21}\sigma_{31}\cdots\sigma_{n1} \in \mathcal{L}(n,1,\{1,\})$. Note that since σ is (123)-avoiding, the largest element in σ must appear in position σ_{12}. Simply remove this element to create a permutation in $\mathcal{L}(n,1,\emptyset)$.

Example 2. Let $n=9, k=1$ and $I=\{1\}$. Consider the permutation $\sigma \in \mathcal{L}(9,1,\{1\})$ given by:

$$\sigma = \underbrace{9\ 10}\ \underbrace{8}\ \underbrace{7}\ \underbrace{5}\ \underbrace{4}\ \underbrace{2}\ \underbrace{6}\ \underbrace{3}\ \underbrace{1}$$

We can remove the 10 to form the permutation $\tau = 987542631$ in $\mathcal{L}(n,1,\emptyset)$.

We can now extend this idea to any index set of size one.

Theorem 6. *There is a bijection between $\mathcal{L}(n,1,\{1\})$ and $\mathcal{L}(n,1,\{k\})$.*

Proof. Let $\sigma = \sigma_{11}\sigma_{12}\sigma_{21}\sigma_{31}\cdots\sigma_{n1} \in \mathcal{L}(n,1,\{1,\})$. First note that σ is (123)-avoiding and $\sigma_{11} < \sigma_{12}$ by definition. We then define the bijection iteratively. At step 1, to create $\sigma^2 \in \mathcal{L}(n,1,\{2,\})$ compare σ_{11} to σ_{21}.

- If $\sigma_{11} > \sigma_{21}$, swap σ_{12} and σ_{21}. I.e., move σ_{12} to the position immediately to the **right** of σ_{21}. Since σ is (123)-avoiding, we know that $\sigma_{21} < \sigma_{12}$, thus the resulting permutation is in $\mathcal{L}(n,1,\{2\})$.
- If $\sigma_{11} < \sigma_{21}$, swap σ_{11} and σ_{12}. I.e., move σ_{11} to the position immediately to the **left** of σ_{21}. Since $\sigma_{11} < \sigma_{21}$, the resulting permutation is in $\mathcal{L}(n,1,\{2\})$.

Call the resulting permutation σ^2.

In general, if σ^i is a permutation in $\mathcal{L}(n,1,\{i\})$, given as

$$\sigma^i = \sigma^i_{11}\sigma^i_{21}\cdots\sigma^i_{(i-1)1}\sigma^i_{i1}\sigma^i_{i2}\sigma^i_{(i+1)1}\cdots\sigma^i_{n1},$$

then to create a permutation in $\mathcal{L}(n,1,\{i+1\})$ do the following:

- If $\sigma^i_{i1} > \sigma^i_{(i+1)1}$, swap σ^i_{i2} and $\sigma^i_{(i+1)1}$. I.e., move σ^i_{i2} to the position immediately to the **right** of $\sigma^i_{(i+1)1}$. Since σ^i is (123)-avoiding, we know that $\sigma^i_{(i+1)1} < \sigma^i_{i2}$, thus the resulting permutation is in $\mathcal{L}(n,1,\{i+1\})$.
- If $\sigma^i_{i1} < \sigma^i_{(i+1)1}$, swap σ^i_{i1} and σ^i_{i2}. I.e., move σ^i_{i1} to the position immediately to the **left** of $\sigma^i_{(i+1)1}$. Since $\sigma^i_{i1} < \sigma^i_{(i+1)1}$, the resulting permutation is in $\mathcal{L}(n,1,\{i+1\})$.

$\square$

Example 3. Let $n=9, k=1$ and $I=\{4\}$. Consider the permutation $\sigma \in \mathcal{L}(9,1,\{4\})$ given by:

$$\sigma = \underbrace{9}\ \underbrace{8}\ \underbrace{7}\ \underbrace{5\ 10}\ \underbrace{4}\ \underbrace{2}\ \underbrace{6}\ \underbrace{3}\ \underbrace{1}$$

Since $\sigma_{41} = 5 > 4 = \sigma_{51}$, then move $\sigma_{42} = 10$ to the right of $\sigma_{51} = 4$, resulting in the permutation $\sigma' \in \mathcal{L}(9,1,\{5\})$ given by:

$$\sigma' = \underbrace{9}\ \underbrace{8}\ \underbrace{7}\ \underbrace{5}\ \underbrace{4\ 10}\ \underbrace{2}\ \underbrace{6}\ \underbrace{3}\ \underbrace{1}$$

Now consider the more general case when k is larger than 1.

Theorem 7. *There is a bijection between $\mathcal{L}(n,k,\{i\})$ and $\mathcal{L}(n,k,\{i+1\})$.*

In general, if σ^i is a permutation in $\mathcal{L}(n,k,\{i\})$ let's consider the ith and the $(i+1)$st blocks, which look like:

$$\sigma_{i1}\sigma_{i2}\cdots\sigma_{ik}\sigma_{i(k+1)}\sigma_{(i+1)1}\sigma_{(i+1)2}\cdots\sigma_{(i+1)k}.$$

To create a permutation in $\mathcal{L}(n,k,\{i+1\})$ do the following:

- If $\sigma_{i1} < \sigma_{(i+1)1}$, move σ_{i1} to the position immediately to the **left** of $\sigma_{(i+1)1}$.
- If $\sigma_{i1} > \sigma_{(i+1)1}$, compare σ_{i2} and $\sigma(i+1)2$. If $\sigma_{i2} < \sigma_{(i+1)2}$ the move σ_{i2} to the position immediately to the left of $\sigma_{(i+1)2}$. Since the original permutation avoids $(123\cdots(k+1)(k+2)$, $\sigma_{(i+1)1} < \sigma_{i2}$, thus the resulting permutation is in $\mathcal{L}(n,1,\{i+1\})$.
- Continue this process inductively until some element of the ith block is moved to the $(i+1)$st block or $\sigma_{il} > \sigma_{(i+1)l}$ for all $1 \leq l \leq k$. In this case, move $\sigma_{i(k+1)}$ to the position immediately to the right of $\sigma_{(i+1)k}$.

To reverse the bijection, begin with a permutation in $\mathcal{L}(n,k,\{i+1\})$ and do the following:

- If $\sigma_{(i+1)(k+1)} < \sigma_{ik}$, move $\sigma_{(i+1)(k+1)}$ to the position immediately to the **right** of σ_{ik}.
- If $\sigma_{(i+1)(k+1)} < \sigma_{ik}$, compare $\sigma_{(i+1)k}$ and σik. If $\sigma_{(i+1)k} < \sigma_{ik}$ and $\sigma_{(i+1)k} > \sigma_{i(k-1)}$ then move $\sigma_{(i+1)k}$ to the position immediately to the left of σ_{ik}.
- If $\sigma_{(i+1)(k+1)} > \sigma_{ik}$ or if $\sigma_{(i+1)(k+1)} < \sigma_{ik}$ and $\sigma_{(i+1)k} < \sigma_{i(k-1)}$, then compare $\sigma(i+1)(k-1)$ and $\sigma i(k-1)$ as above. Continue this process inductively until some element of the $(i+1)st$th block is moved to the ith block or $\sigma_{il} > \sigma_{(i+1)l}$ for all $1 \leq l \leq k$.

Example 4. Let $n=5, k=2$, $I=\{1,2\}$ and $I'=\{1,3\}$. Consider the permutation $\sigma \in \mathcal{L}(5,2,\{1,3\})$ given by:

$$\sigma = \underbrace{6\ 10\ 12}\ \underbrace{2\ 8\ 11}\ \underbrace{1\ 9}\ \underbrace{5\ 7}\ \underbrace{3\ 4}$$

Since $\sigma_{21} = 2 > 1 = \sigma_{31}$, then compare σ_{22} and σ_{32}. Since $\sigma_{22} = 8 < 9 = \sigma_{32}$ then move $\sigma_{22} = 8$ to the left of $\sigma_{32} = 9$, resulting in the permutation $\sigma' \in \mathcal{L}(5,2,\{1,3\})$ given by:

$$\sigma' = \underbrace{6\ 10\ 12}\ \underbrace{2\ 11}\ \underbrace{1\ 8\ 9}\ \underbrace{5\ 7}\ \underbrace{3\ 4}$$

Theorem 8. *For any I and I' for which $|I| = |I'|$, there is a bijection between $\mathcal{L}(n,k,I)$ and $\mathcal{L}(n,k,I')$*
.

Proof. The proof follows directly by utilizing the bijection in Proposition 3.3 both directly and inversely to change I into I'. $\square$

Note, there is some choice of which order in which to convert extended blocks in I into extended blocks in I' and it can be shown that the resulting permutation is independent of the order in which blocks are converted. In general, one begins with the largest block of I and tries to convert it to the largest block of I'. If, however, there is another block of I in between these largest blocks, then one finds the block of I closest to the largest block of I' and converts this one. The process is similar to the process of converting a permutation in S_n to another permutation in S_n utilizing only adjacent element swaps. A general method which works (but may not be the most efficient!) is to convert a permutation in $\mathcal{L}(n,k,I)$ with $|I| = j$ to one in $\mathcal{L}(n,k,\{1,2,\cdots j\})$ and then to convert that permutation into one in $\mathcal{L}(n,k,I')$. One could similarly, of course, convert a permutation in $\mathcal{L}(n,k,I)$ into one where I is the subset of the largest j blocks and then convert that into a permutation in $\mathcal{L}(n,k,I')$.

Example 5. Let's continue with the same example as above.
Let $n=5, k=2$, $I=\{1,2\}$ and $I'=\{2,5\}$ and consider the same permutation $\sigma \in \mathcal{L}(5,2,\{1,3\})$ given by:

$$\sigma = \underbrace{6\ 10\ 12}\ \underbrace{2\ 8\ 11}\ \underbrace{1\ 9}\ \underbrace{5\ 7}\ \underbrace{3\ 4}$$

In the example above, we created the permutation $\sigma' \in \mathcal{L}(5,2,\{1,3\})$ given by:

$$\sigma' = \underbrace{6\ 10\ 12}\ \underbrace{2\ 11}\ \underbrace{1\ 8\ 9}\ \underbrace{5\ 7}\ \underbrace{3\ 4}$$

By comparing appropriate elements of the third and fourth blocks, we create the permutation $\sigma^{(2)} \in \mathcal{L}(5,2,\{1,4\})$ given by:

$$\sigma' = \underbrace{6\ 10\ 12}\ \underbrace{2\ 11}\ \underbrace{8\ 9}\ \underbrace{1\ 5\ 7}\ \underbrace{3\ 4}$$

and then comparing the elements of the fourth and fifth blocks, we create the permutation $\sigma^{(3)} \in \mathcal{L}(5,2,\{1,5\})$ given by:

$$\sigma' = \underbrace{6\ 10\ 12}\ \underbrace{2\ 11}\ \underbrace{8\ 9}\ \underbrace{5\ 7}\ \underbrace{1\ 3\ 4}$$

. Now compare elements of the first and second blocks to create the permutation τ in $\mathcal{L}(5,2,\{2,5\})$ given by:

$$\sigma' = \underbrace{6\ 12}\ \underbrace{2\ 10\ 11}\ \underbrace{8\ 9}\ \underbrace{5\ 7}\ \underbrace{1\ 3\ 4}$$

.

Theorem 9. *For any I and I', there is a bijection between $\mathcal{L}(n,k,I)$ and $\mathcal{L}(n,k,I')$.*

Proof. The proof follows by utilizing Theorem 3.4 and the bijection given by Mei and Wang [1] from $\mathcal{L}(n,1,\{1\})$ and $\mathcal{L}(n,1,\emptyset)$. If $|I| < |I'|$ then first convert the extended blocks of I into the largest blocks of I', then add largest elements into the first block as per the bijection of Mei and Wang. Once a new largest element has been added to the first block, convert the extended first block into the largest block of I' that has not yet been extended. If $|I| > |I'|$, first convert the smallest block of I' into an extended first block and remove the largest element as per the bijection of Mei and Wang. Continue to do this until $|I| = |I'|$, then utilize the bijection given above. $\square$

References

[1] Zhousheng Mei, Suijie Wang, *Pattern Avoidance and Young Tableaux* Electronic Journal of Combinatorics **24(1)** (2017), 6 pps.

[2] Joel Brewster Lewis, *Pattern Avoidance for Alternating Permutations and Young Tableaux* J. Combin. Theory Ser. A **110** (2011), 1436-1450.