Mesh Pattern Poset

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Abstract

We introduce the mesh pattern poset, that is, the poset of all mesh patterns ordered by containment. This poset contains the classical permutation poset as an induced subposet. We introduce some preliminary results on the purity, Möbius function, topology and structure of this poset.

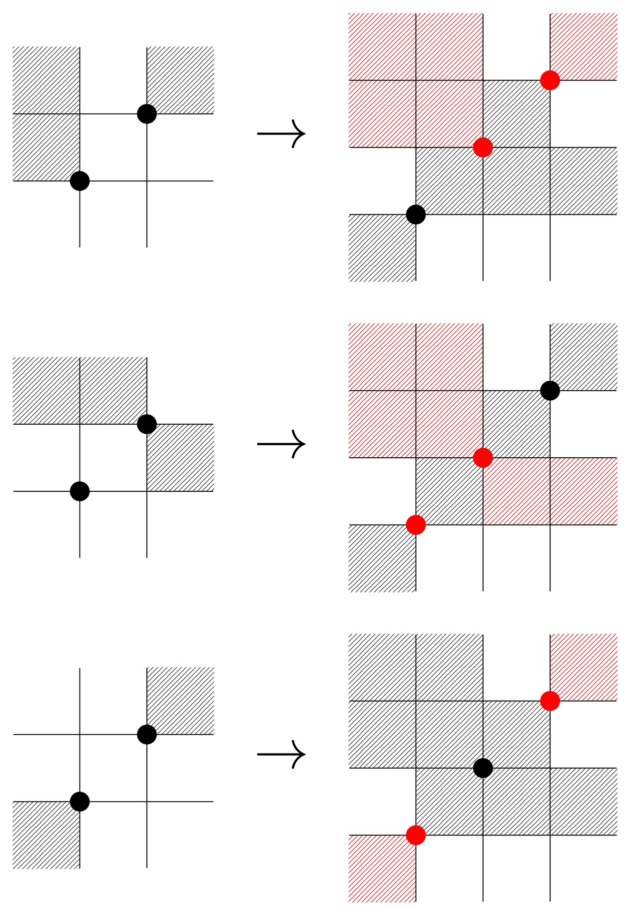
Main Results

1. Full classification of which intervals $[\cdot, m]$ are pure.
2. Möbius function is unbounded, but $\mu(\cdot, m)$ is almost always zero.
3. The topology and Möbius function of $[(\sigma, S), (\pi, P)]$ is not dictated by $[\sigma, \pi]$.
4. There are an infinite number of maximal elements, the fully shaded mesh patterns.

Mesh Pattern Poset

An occurrence of (σ, S) in (π, P) is an occurrence of σ in π, where the area of (π, P) corresponding to a shaded box of (σ, S) is fully shaded and contains no points.
The mesh pattern poset contains all mesh patterns with:

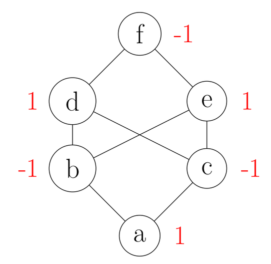
$$(\sigma, S) \leq (\pi, P) \text{ if } (\sigma, S) \text{ occurs in } (\pi, P)$$



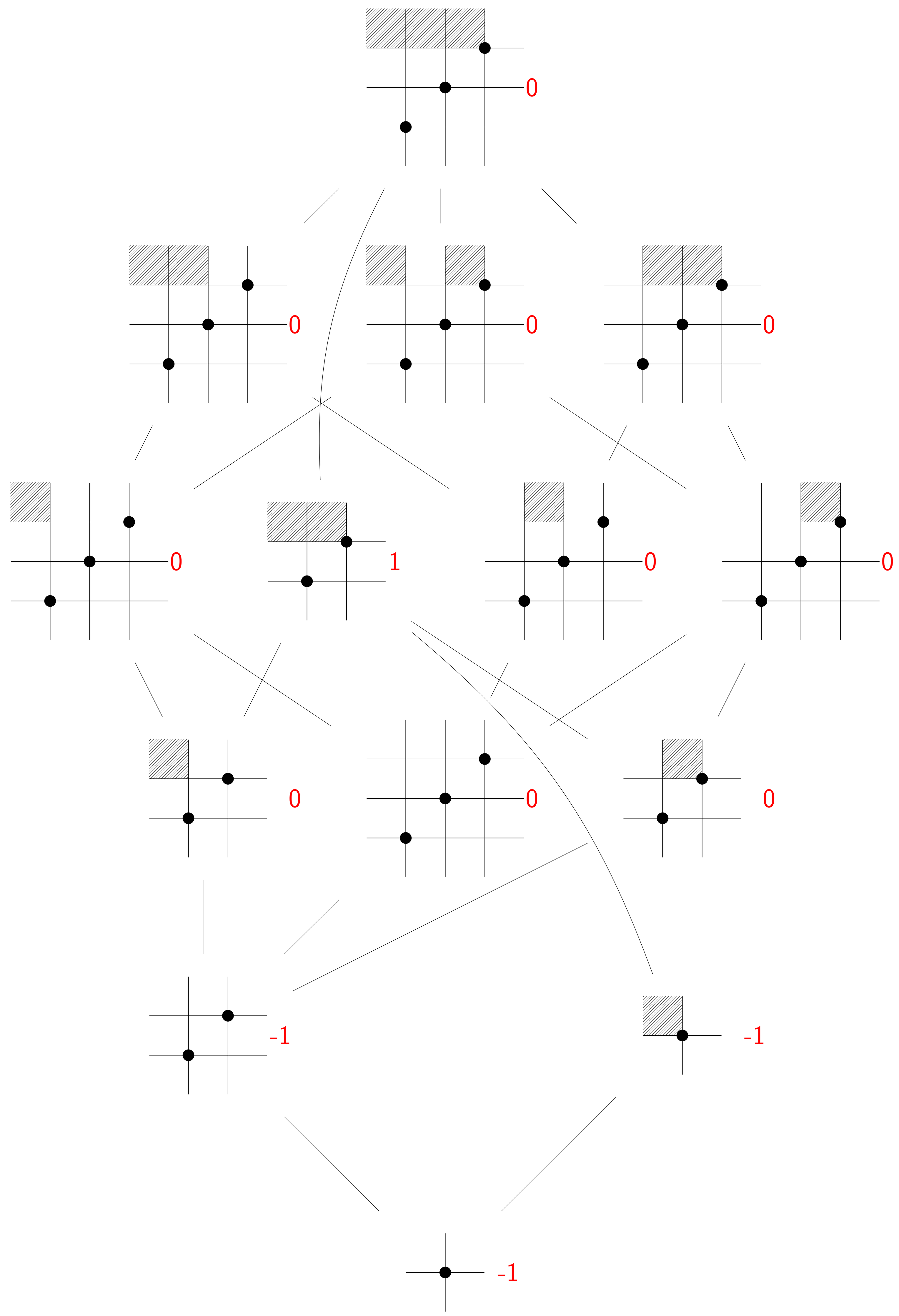
Definitions

Pure: Every maximal chain has the same length.

Möbius function: $\mu(a, a) = 0$, $\mu(a, b) = -\sum_{c \in [a,b)} \mu(a, c)$.



$\mu(a, f) = -1$



Purity

The interval $[\cdot, m]$ is non-pure if and only if there exists a point p in m, where deleting p gives the mesh pattern x and occurrence η of x in m satisfying:

1. Deleting p merges shadings.
2. There is no occurrence of x in m where the shaded boxes are a subset of those in η.

A non-pure and pure interval, with p circled, η in red and x beneath

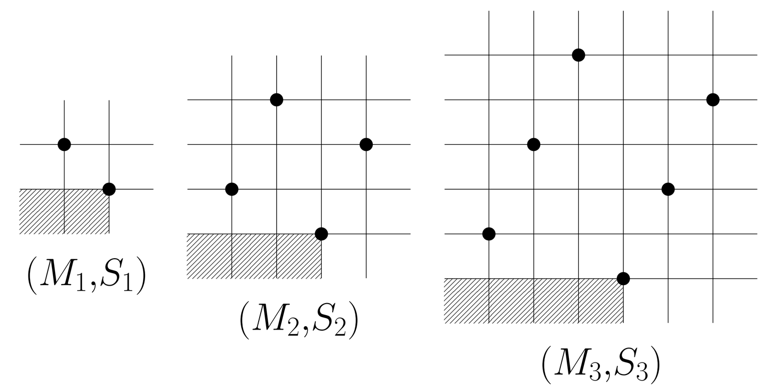
Möbius function

1. $[\sigma, \pi] \approx [(\sigma, \emptyset), (\pi, \emptyset)] \implies \mu(\sigma, \pi) = \mu((\sigma, \emptyset), (\pi, \emptyset))$

   The Möbius function is unbounded on the permutation poset
   $\Downarrow$
   The Möbius function is unbounded on the mesh pattern poset.

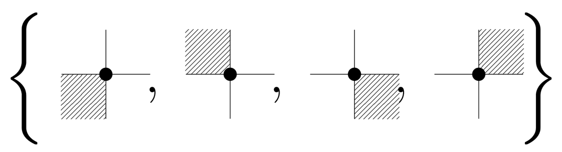
2. Möbius function is unbounded on mesh patterns with shadings:

$$M_m = 246 \ldots (2m)135 \ldots (2m-1)$$
$$S_m = \{(0,0), (1,0), \ldots, (n+1, 0)\}$$



$|\mu(M_1, M_m)| = m$ and $[M_1, M_m]$ is shellable.

Proof: Isomorphic to interval of subword poset.

3. If m does not contain any of , then

$$\mu(\cdot, m) = 0$$

Therefore, for almost all mesh patterns $\mu(\cdot, m) = 0$,

that is, as $|m| \to \infty$ the probability $\mu(\cdot, m) = 0$ tends to 1.