ON SUPER-STRONG WILF EQUIVALENCE CLASSES OF PERMUTATIONS

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MOTIVATION

Is the number of elements of the symmetric group $\mathcal{S}_n$ on n letters that are Wilf equivalent to a given permutation always a power of 2?

INTRODUCTION

Let $\mathbb{P}$ be the set of positive integers. For a word $w \in \mathbb{P}^*$, let $|w|$ be its length and $\|w\|$ be its height, i.e. the sum of its letters. The *weight* of w is the monomial $t^{|w|}x^{\|w\|}$. Suppose $u, w \in \mathbb{P}^*$. An embedding of u in w is a factor v of w such that $|v| = |u| = k$ and $u_i \leq v_i$ for all $i \in [k]$. If the first element of v is the j-th element of w then the index j is called an *embedding index* of u into w. The *embedding set* $Em(u, w)$ is the set of all embedding indices of u into w.

- **Wilf equivalence** $u \sim v \Leftrightarrow$ they have the same weight-generating function.
- **Strong Wilf equivalence** $u \sim_s v \Leftrightarrow \exists$ a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ s.t. $|Em(u, w)| = |Em(v, f(w))|, \forall\, w \in \mathbb{P}^*$.
- **Super-strong Wilf equivalence** $u \sim_{ss} v \Leftrightarrow \exists$ a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ s.t. $Em(u, w) = Em(v, f(w)), \forall\, w \in \mathbb{P}^*$.

CLUSTER METHOD

Given a word $u \in \mathbb{P}^*$ and a set E, the *minimal cluster* $m(u, E)$ of u with embedding set E is the unique word w s.t. $Em(u, w) = E$ and none of the entries of w can be decreased without altering the set of its factors.

Minimal Cluster Rearrangement Theorem (MCRT): Let $u, v \in \mathbb{P}^*$. Then $u \sim_{ss} v \Longleftrightarrow m(u, E)$ and $m(v, E)$ are rearrangements of one another, for every embedding set E.

Intersection Rule: Let $u, v \in \mathcal{S}_n$ and $s = u^{-1}, t = v^{-1}$. Then

$$u \sim_{ss} v \Longleftrightarrow \left|\overline{s_i} \cap \Big(\bigcup_{j=i+1}^{n} \overline{s_j}\Big)\right| = \left|\overline{t_i} \cap \Big(\bigcup_{j=i+1}^{n} \overline{t_j}\Big)\right|$$

for each $i \in [n-1]$ and every embedding set E, where $\overline{k}$ is the shift of E by $k-1$ positions to the right.

SEQUENCE OF DIFFERENCES

Let $u, v \in \mathcal{S}_n$. We say that u is *cross equivalent* to v, i.e. $u \sim_+ v$, if $i^+(u) = i^+(v), \forall\, i \in [n-1]$, where $i^+(u)$ is defined as the multiset of distances

$$i^+(u) = \{d_u(i, j) : j \in [i+1, n]\}.$$

We have shown that super-strong Wilf equivalence is a strict refinement of cross equivalence.

Given a permutation u, its inverse $s = u^{-1}$ and a letter i, the vector of consecutive differences $\Delta_i(s)$ for $i \in [2, n-1]$, contains the distances between letters in u that are greater than or equal to i as they appear sequentially in u from left to right.

Theorem. Let $u, v \in \mathcal{S}_n$, $s = u^{-1}$ and $t = v^{-1}$. Then

$$u \sim_{ss} v \Longleftrightarrow \Delta_i(s) = \Delta_i(t), \forall i \in [2, n-1].$$

TREE REPRESENTATION

We denote by $T^n(u)$ the ordered rooted tree whose leaves constitute the cross equivalence class of u. The nodes of $T^n(u)$ are defined to be the partly filled words

$$S_i^n = \{x \in A^n : |x|_j = 1 \text{ for } 1 \leq j \leq i \text{ and } |x|_* = n-i\},$$

where $A = [n] \cup \{*\}$, $i \in [0, n]$, and $|x|_a$ is the number of occurrences of the letter a in the word x. The elements at the i-th level of $T^n(u)$ constitute the set

$$L_i^n(u) = \{x \in S_i^n \;:\; d_x(i, *) = i^+(u)\}.$$

The word $y \in S_{i+1}^n$ is a child of the word $x \in S_i^n$ if they have exactly $n-1$ letters in common.

Proposition. $T^n(u)$ is a binary tree, where at each level the number of children is the same throughout all nodes and is either equal to 1 or 2.

For a word $x \in S_i^n$, let $f(x)$ be the factor of x whose first and last letter is respectively the first and last $*$ that appear in x. Let $c(x)$ be the configuration word in letters $\{*, \circ\}$ that we obtain if we replace all numbers in $f(x)$ with $\circ$.

A vertex $x \in T^n(u)$ that has two children y and y' is labeled 0 if $c(y) = c(y')$, and 1 otherwise.

Theorem. Let $u, v \in \mathcal{S}_n$. Suppose that $u \sim_+ v$. Then $u \sim_{ss} v$ if and only if one can get from u to v in $T_n(u)$ by following a path that avoids switching direction (from left to right or vice-versa) on vertices at the same level which are labeled 1.

Corollary. Let $u \in \mathcal{S}_n$ and let k, l be the number of levels in $T^n(u)$ labeled 0 and 1, respectively.
- The number of words in each super-strong Wilf equivalence class in $T^n(u)$ is equal to 2^k.
- The class $[u]_+$ is partioned into 2^l distinct super-strong Wilf equivalence classes.

EXAMPLE

Let $u = 21365874$. Then $s = u^{-1} = 21385476$. The sequence of differences for s is the following:

$$\Delta_7(s) = (1)$$
$$\Delta_6(s) = (2, 1)$$
$$\Delta_5(s) = (1, 1, 1)$$
$$\Delta_4(s) = (1, 1, 1, 1)$$
$$\Delta_3(s) = (1, 1, 1, 1, 1)$$
$$\Delta_2(s) = (2, 1, 1, 1, 1, 1)$$

The multisets of distances for the word u are:

$7^+(u) = \{1\}$, $6^+(u) = \{2, 3\}$,
$5^+(u) = \{1, 1, 2\}$, $4^+(u) = \{1, 2, 3, 4\}$,
$3^+(u) = \{1, 2, 3, 4, 5\}$, $2^+(u) = \{2, 3, 4, 5, 6, 7\}$,
$1^+(u) = \{1, 1, 2, 3, 4, 5, 6\}$

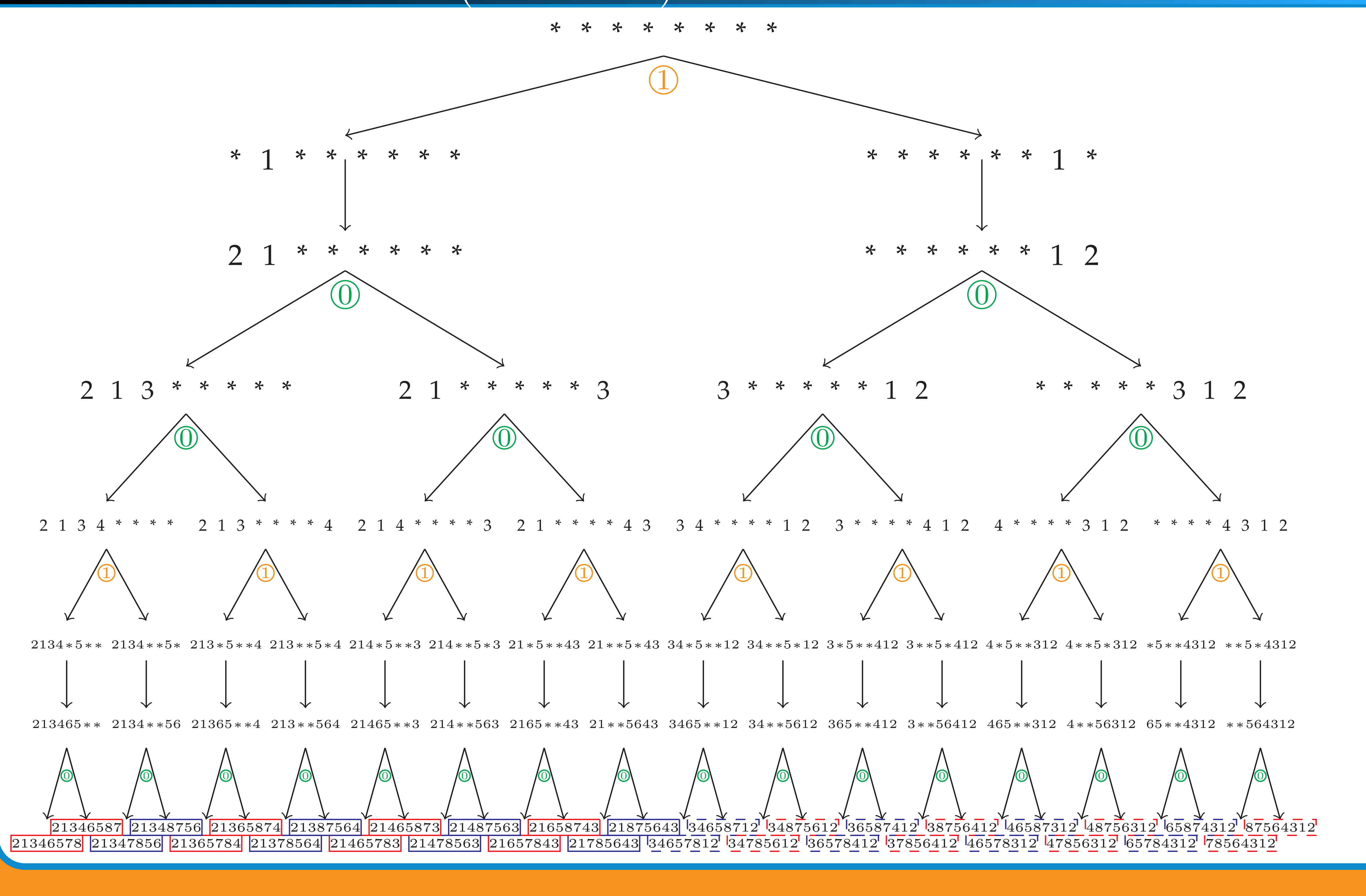
The binary tree $T^8(u)$ is shown below. The cross equivalence class of u, that is represented by the leaves of $T^8(u)$, is partitioned in the following four super-strong Wilf equivalence classes, where $v = 21347856$ and $\tilde{u}$ and $\tilde{v}$ denote the reversals of u and v respectively.

	Class
u	21346578, 21346587, 21365784, 21365874, 21465783, 21465873, 21657843, 21658743
v	21347856, 21348756, 21378564, 21387564, 21478563, 21487563, 21785643, 21875643
$\tilde{u}$	34785612, 34875612, 37856412, 38756412, 47856312, 48756312, 78564312, 87564312
$\tilde{v}$	34657812, 34658712, 36578412, 36587412, 46578312, 46587312, 65784312, 65874312

OPEN PROBLEMS

- Investigate further connections amongst Wilf, cross, shift and super-strong Wilf equivalence.
- Enumerate all cross equivalence and super-strong Wilf equivalence classes for a given $n \in \mathbb{N}$.

THE BINARY TREE $T^8(21365874)$



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