Elmsley's Problem for horseshoe permutations

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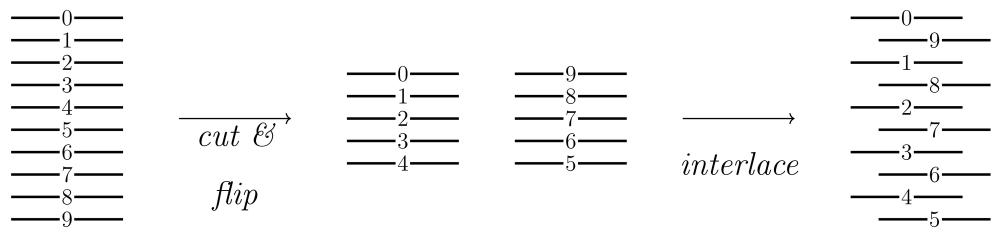
Horseshoe Shuffles and Elmsley's Problem

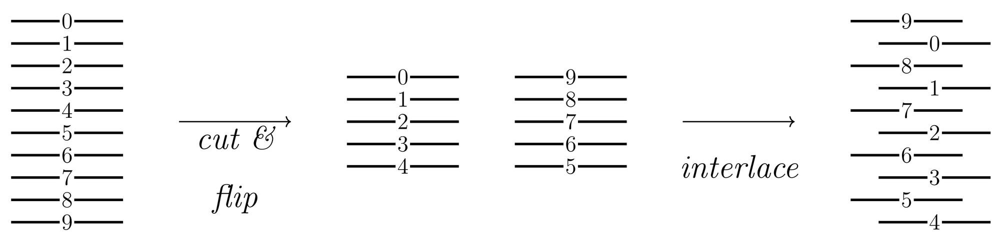
Definition 1. *To perform a* ***horseshoe shuffle*** *on a deck with* $2n$ *cards:*

1. *split the deck perfectly in half,*
2. *reverse the order of the cards in the second half, and*
3. *interlace perfectly the two halves.*

Out horseshoe shuffle*: top card stays on top*



In horseshoe shuffle*: top card does* not *stay on top*



Elmsley's problem. Find all minimal sequences of horseshoe shuffles that move a given card to the top of the deck.

Deck of size 2^r

Theorem 1 (Butler–Diaconis–Graham [1])**.** *Every card in a deck of* 2^r *cards has a unique Elmsley sequence.*

If we encode the positions as binary words $x_{r-1}\cdots x_0$, then we have

$$\textit{out}\text{ shuffle}: x_{r-1}x_{r-2}...x_1x_0 \longrightarrow \begin{cases} x_{r-2}...x_1x_0 0 & \text{if } x_{r-1}=0 \\ \overline{x_{r-2}\cdots x_1x_0}1 & \text{if } x_{r-1}=1 \end{cases}$$

$$\textit{in}\text{ shuffle}: x_{r-1}x_{r-2}...x_1x_0 \longrightarrow \begin{cases} x_{r-2}...x_1x_0 1 & \text{if } x_{r-1}=0 \\ \overline{x_{r-2}...x_1x_0}0 & \text{if } x_{r-1}=1 \end{cases}$$

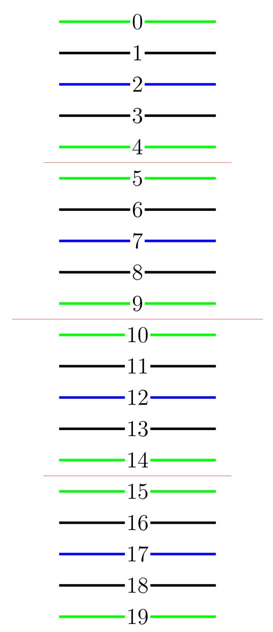
Algorithm 1. *This leads to an algorithm to compute the* unique *Elmsley sequence for every card.*

	$x_{r-1}=0$	$x_{r-1}=1$
$x'_0=0$	out	in
$x'_0=1$	in	out

$$\begin{array}{cccc|l} 1 & 0 & 0 & 1 & \\ & & & & \textit{in} \\ 1 & 1 & 0 & y_1 & \\ & & & & \textit{out} \\ 0 & 1 & \overline{y_1} & y_2 & \\ & & & & \textit{in} \\ 1 & \overline{y_1} & y_2 & y_3 & \\ & & & & \textit{in} \\ y_1 & \overline{y_2} & \overline{y_3} & y_4 & \\ 0 & 0 & 0 & 0 & \end{array} \qquad \begin{array}{cccc|l} 0 & 1 & 1 & 1 & \\ & & & & \textit{in} \\ 1 & 1 & 1 & y_1 & \\ & & & & \textit{in} \\ 0 & 0 & \overline{y_1} & y_2 & \\ 0 & 0 & 0 & 0 & \end{array}$$

Figure 1: The table indicates the shuffle to perform depending on the first letter of the binary word (x_{r-1}) and the last letter of the binary word we want to obtain (x'_0). Then, we use the table to compute the unique Elmsley sequence of the card at position 9 and the unique Elmsley sequence of the card at position 7 in a deck of 16 cards.

Boundaries, boundary cards and middle cards



Assume a deck of $2n = 2^r m$ cards with $m > 1$ odd.

Definition 2. *The* order 1 boundary *is the separation of the deck into two halves that we call* subdecks (of order 1). *The* order i boundaries *for* $2 \leqslant i \leqslant r$ *are the separations of the order* $i-1$ *subdecks into two equal parts.*

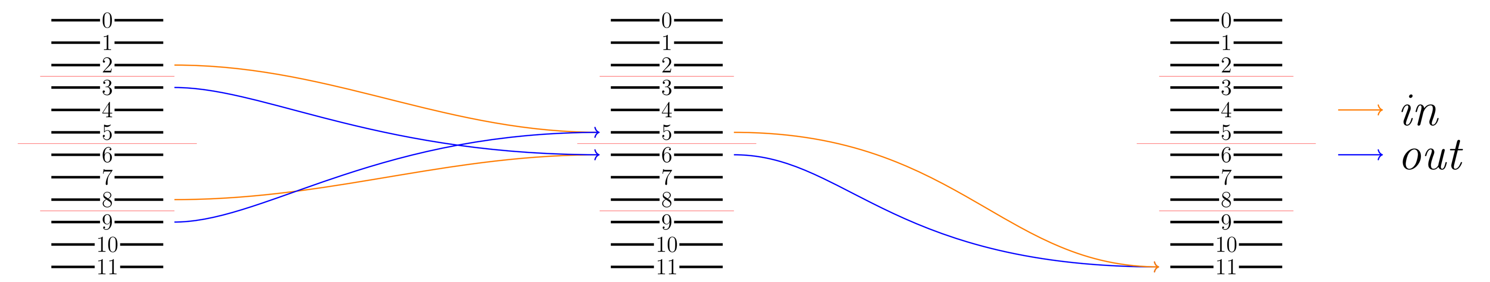
Definition 3. *A card is a* boundary card *if it is directly above or below a boundary, or if it is the top or bottom card.*

Definition 4. *A* middle card *is a card in the center of one of the smallest subdecks defined by the boundaries.*

Boundary cards

The *in* shuffle always brings the last card to the top of the deck while the *out* fixes the top card. Hence, to get on top, a card needs to get to the last position first. Similarly, to go to the last position a card must pass through the order 1 boundary.

Proposition 1. *The* out *shuffle sends the card which is under an order* i *boundary to an order* $i-1$ *boundary. The* in *shuffle sends the card which is above an order* i *boundary to an order* $i-1$ *boundary. Also, after a shuffle, all cards at order* $i-1$ *boundaries came from the same side of order* i *boundaries.*



Theorem 2 (N.–S. [2, 3])**.** *Every boundary card has a unique Elmsley sequence. If the boundary card is of order* i*, then its Elmsley sequence is of length* $i+1$*.*

Algorithm 2. *The Elmsley sequence of a boundary card is obtained by applying the* 2^r*-algorithm to the deck consisting only of the boundary cards (i.e., the deck obtained by removing the non-boundary cards).*

Middle cards

Remark that both the *in* and *out* horseshoe shuffles move middle cards to the highest possible order boundaries, and that all such boundary cards are obtained this way. This implies that the Elmsley sequences of all non-boundary cards are not unique.

Theorem 3 (N.–S. [2, 3])**.** *A card has a unique Elmsley sequence iff it is a boundary card. (All cards in a deck of size* 2^r *are boundary cards.)*

Other cards

Theorem 4 (N.–S. [2, 3])**.** *For each non-boundary card, the shortest sequence of* in *and* out *horseshoe shuffles that moves it to a middle card position is unique. Consequently, all non-boundary cards have exactly two Elmsley sequences.*

Build a tree T_m starting with root $\frac{m-1}{2}$ and recursively applying the following rules:

① $\frac{p}{2}$ (came from *inverse out* on an even position)

② $\frac{p-1}{2}$ (came from *inverse in* on an odd position)

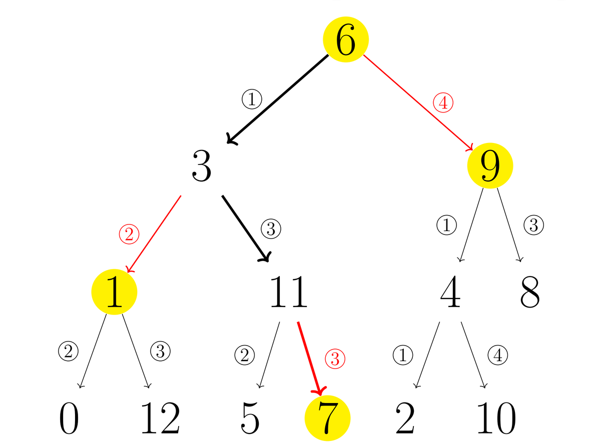
③ $(m-1) - \frac{p-1}{2}$ (came from *inverse out* on an odd position)

④ $(m-1) - \frac{p}{2}$ (came from *inverse in* on an even position)

Trim the tree to keep only the first appearance of integers; call this finite tree $\widetilde{T_m}$.

Algorithm 3. *To compute the sequence to a middle position of a card* x *we then proceed as follow:*

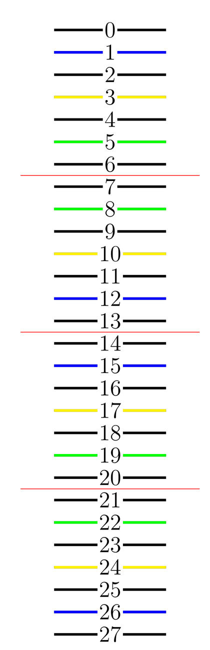
1. *Compute* $a = x \pmod m$.
2. *Determine the next shuffle* S*: if the incoming edge of* a *in* $\widetilde{T_m}$ *is labeled* ① *or* ③ *the shuffle is an* out shuffle*, otherwise it's an* in shuffle.
3. *Add* S *to the sequence.*
4. *If* $S(x)$ *is not a middle position, repeat with* $x \leftarrow S(x)$.



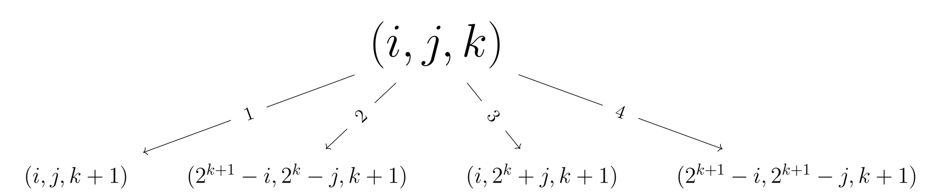
- $\textit{out}(7) = 14$
- $\textit{in}(14) = 22$
- $\textit{in}(22) = 6$

Figure 2: The sequence for card 7 in a 26 card deck is out, in, in.

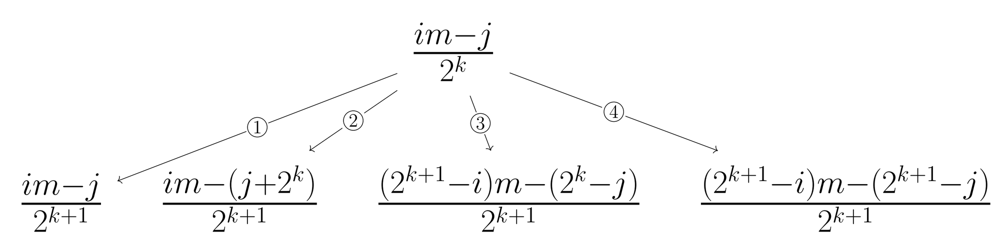
Tools for the proof



Construction rules of T



Construction rules of T_m



References

[1] Steve Butler, Persi Diaconis, and Ron Graham. The mathematics of the flip and horseshoe shuffles. Amer. Math. Monthly, 123(6):542–556, 2016.

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[3] Émile Nadeau and Stéphanie Schanck. Résultats sur le problème d'Elmsley pour les mélanges horseshoe. Rapport technique, Laboratoire de combinatoire et d'informatique mathématique (LaCIM), Université du Québec à Montréal, August 2015.