Quadrant marked mesh patterns in 123-avoiding permutations

Overview

- **Mesh patterns** are extension of classical patterns as statistics of permutations. The notion of **mesh patterns** was introduced by Brändén and Claesson.
- Kitaev and Remmel initiated the systematic study of distribution of **quadrant marked mesh patterns** on permutations.
- This study was extended to 132-**avoiding permutations** by Kitaev, Remmel and Tiefenbruck.
- **Our goal: to study the distribution of quadrant marked mesh patterns in** 123-**avoiding permutations.**

Preliminary

Classical Permutation Patterns

- **red**$[w]$ Given a sequence $w = w_1 \dots w_n$ of distinct integers, red$[w]$ is the permutation found by replacing the i^{th} largest integer that appears in σ by i.
  Example. If $w = 4592$, then red$[w] = 2341$.
- Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say that the pattern τ **occurs** in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ provided there exists $1 \le i_1 < \cdots < i_j \le n$ such that red$[\sigma_{i_1} \dots \sigma_{i_j}] = \tau$. We say that a permutation σ **avoids** the pattern τ if τ does not occur in σ.
- Let $\mathcal{S}_n(\tau)$ denote the set of permutations in $\mathcal{S}_n$ which avoid τ.

Quadrant Marked Mesh Patterns

- The graph of $\sigma = \sigma_1 \dots \sigma_n$, $G(\sigma)$, is the set of points (i, σ_i) for $i = 1, \dots, n$.
  Example.

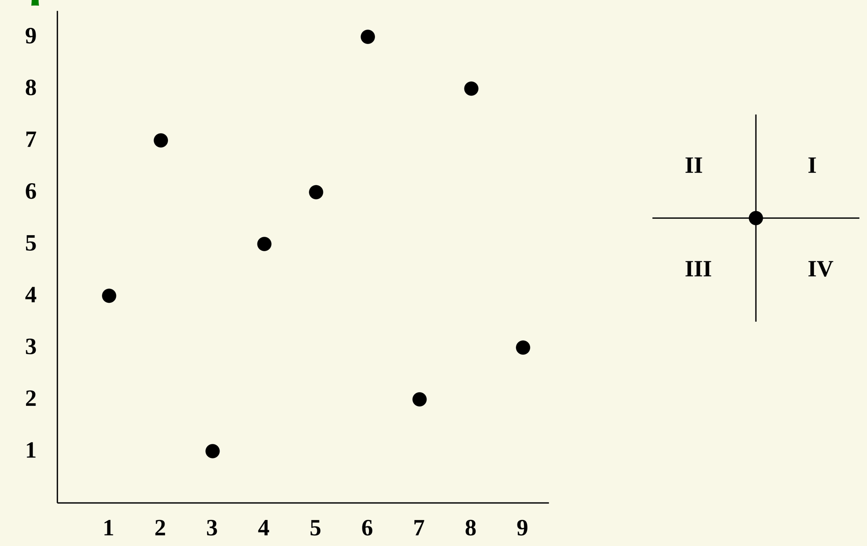
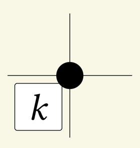
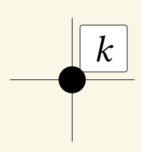
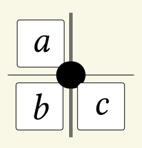
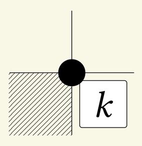


Figure 1: The graph $G(\sigma)$ of $\sigma = 471569283$.

- We say that σ_i in σ matches the **quadrant marked mesh pattern** MMP(a, b, c, d) if taking (i, σ_i) as its origin in $G(\sigma)$, there are $\ge a$ points in quadrant I, $\ge b$ points in quadrant II, $\ge c$ points in quadrant III, and $\ge d$ points in quadrant IV.
  Example. If $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the marked mesh pattern MMP$(2, 1, 2, 1)$.
- We consider MMP(a, b, c, d) where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. A coordinate equaling $\emptyset$ means there is no points in the corresponding quadrant. Using the **(two-dimensional) notation of Úlfarsson for marked mesh patterns**, we have

MMP$(0, 0, k, 0)$ =  , MMP$(k, 0, 0, 0)$ =  ,

MMP$(0, a, b, c)$ =  , MMP$(0, 0, \emptyset, k)$ =  .

- For any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$ and permutation τ, we let $Q^{(a,b,c,d)}_{n,\tau}(x) = \sum_{\sigma \in \mathcal{S}_n(\tau)} x^{\text{mmp}^{(a,b,c,d)}(\sigma)}$ and
$$Q^{(a,b,c,d)}_{\tau}(t, x) = 1 + \sum_{n \ge 1} t^n Q^{(a,b,c,d)}_{n,\tau}(x)$$
- For any a, b, c, d, we write $Q^{(a,b,c,d)}_{n,\tau}(x)|_{x^k}$ for the coefficient of x^k in $Q^{(a,b,c,d)}_{n,\tau}(x)$.

Dyck Path Bijections

- A **down-rignt Dyck path** is a lattice path from $(0, n)$ to $(n, 0)$ which stays on or below the diagonal $x = -y$.
- D_n is the set of $n \times n$ Dyck paths.
- Many of our results are proved by these two bijections:

1. Krattenthaler's bijection, $\Phi : S_n(132) \to D_n$.
2. Elizalde and Deutsch's bijection, $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$.

- The map $\Phi : S_n(132) \longrightarrow D_n$ is defined by mapping the left-to-right minima to peaks of a Dyck path, as shown in Figure 2. The map $\Phi^{-1} : D_n \longrightarrow S_n(132)$ is described by marking the peaks and lowest rows without mark for the columns without a peak from left to right.

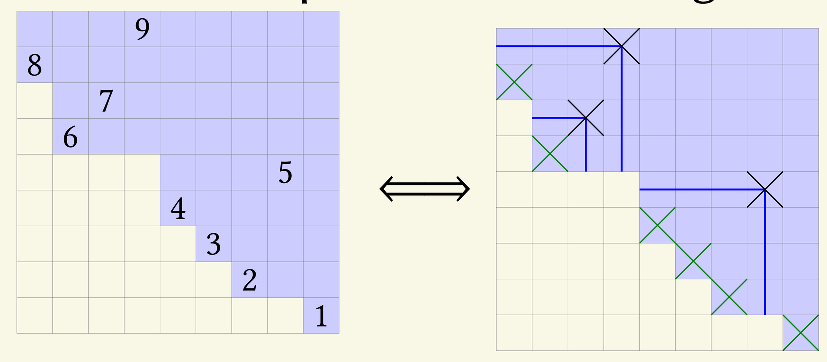


Figure 2: $\mathcal{S}_n(132)$ to $\mathcal{D}_n$

- The map $\Psi : S_n(123) \longrightarrow D_n$ is defined by the exact same process. The map Φ^{-1} is described by marking the peaks, and highest rows without mark for the columns without a peak from left to right.

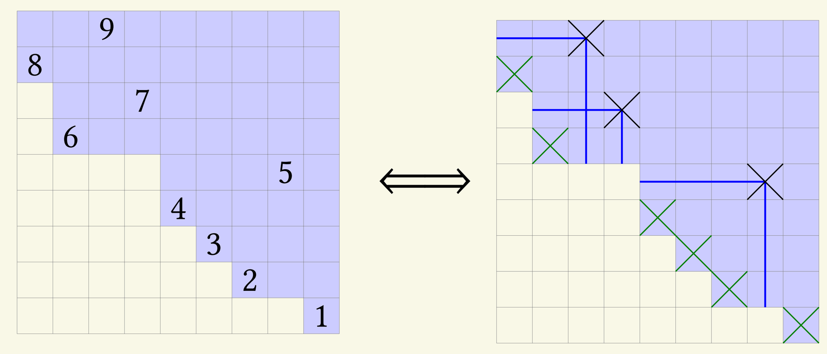


Figure 3: $\mathcal{S}_n(123)$ to $\mathcal{D}_n$

General Results about $Q^{(a,b,c,d)}_{123}(t, x)$

- $\mathcal{S}_n(123)$ is closed under the operation reverse-complement and inversion, Thus
- **Lemma.** or any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$,
$$Q^{(a,b,c,d)}_{n,123}(x) = Q^{(c,d,a,b)}_{n,123}(x)$$
$$= Q^{(c,b,a,d)}_{n,123}(x) = Q^{(a,d,c,b)}_{n,123}(x)$$
- Any 123-avoiding permutation can not have pattern MMP(a, b, c, d) where $a, c \ge 1$, otherwise the permutation contains an occurrence of 123.
- Thus, no permutations $\sigma \in \mathcal{S}_n(123)$ can match MMP(a, b, c, d) where $a, c \ge 1$. If $a \ge 1$, then $Q^{(a,b,0,d)}_{123}(t, x) = Q^{(a,b,\emptyset,d)}_{123}(t, x)$.
- The map $\Psi^{-1} \circ \Phi$ give a bijection between $S_n(132)$ and $S_n(123)$, see Figure 4. This bijection allows us to prove the following theorem.
  Theorem 1. For any $k > 0$ and $\ell, m \ge 0$, $Q^{(k,\ell,0,m)}_{123}(t, x) = Q^{(k,\ell,\emptyset,m)}_{123}(t, x) = Q^{(k,\ell,\emptyset,m)}_{132}(t, x)$.

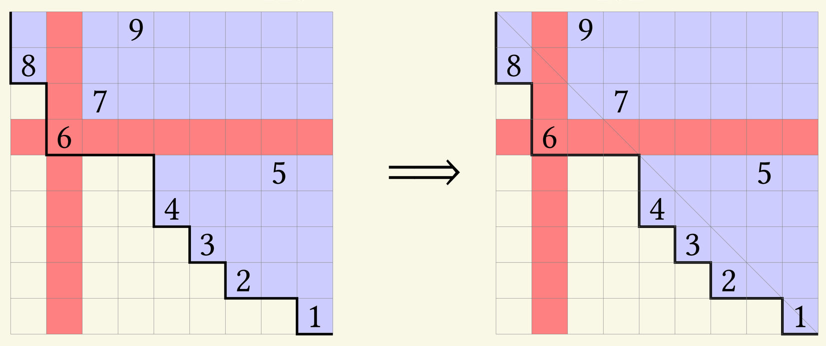


Figure 4: $\mathcal{S}_n(132)$ to $\mathcal{S}_n(123)$ keeps MMP$(k, \ell, \emptyset, m)$

- Thus to compute $Q^{k,\ell,0,m}_{123}(x, t)$ where $k > 0$, we need only compute $Q^{k,\ell,\emptyset,m}_{132}(x, t)$ which can easily be computed using the techniques of Kiteav, Remmel, and Tiefenbruck. They computed $Q^{k,0,0,0}_{132}(x, t)$ where $k \ge 0$.
- We can find the generating function $Q^{k,\ell,0,m}_{132}(x, t)$ for any $k, \ell, m \ge 0$.
- We list a couple of examples of our results below.
  Theorem 2.
$$Q^{(0,0,0,0)}_{132}(t, x) = \frac{1 + t - tx - \sqrt{(1 + t - tx)^2 - 4t}}{2t}.$$
  For $k > 0$,
$$Q^{(0,k,0,0)}_{132}(t, x) = \frac{1 + t\sum_{i=1}^{k-1} C_{i-1}t^{i-1}(Q^{(0,k-i,0,0)}_{132}(t, x) - Q^{(0,0,0,0)}_{132}(t, x))}{1 - tQ^{(0,0,0,0)}_{132}(t, x))}$$
  Theorem 3. For all $k, \ell > 0$,
$$Q^{(k,\ell,0,0)}_{132}(t, x) = 1 + t\sum_{i=1}^{\ell-1} C_{i-1}t^{i-1}Q^{(k,\ell-i,0,0)}_{132}(t, x)$$
$$+(Q^{(k-1,\ell,0,0)}_{132}(t, x) - \sum_{i=0}^{\ell-2} C_i t^i)Q^{(k,0,0,0)}_{132}(t, x).$$
- **Theorem 1** implies that $Q^{(a,b,0,d)}_{n,123}(x)|_{x^k} = Q^{(0,d,a,b)}_{n,123}(x)|_{x^k} = Q^{(a,b,0,d)}_{n,132}(x)|_{x^k}$.
- About the coefficients of x^0 and x^1 in functions $Q^{(a,b,0,d)}_{n,132}(x)$, we prove that
  Theorem 4. $Q^{(k,\ell,\emptyset,m)}_{n,132}(x)\big|_{x^0} = Q^{(k,\ell,0,m)}_{n,132}(x)\big|_{x^0}$ and $Q^{(k,\ell,\emptyset,m)}_{n,132}(x)\big|_{x^1} = Q^{(k,\ell,0,m)}_{n,132}(x)\big|_{x^1}$.
- The reason that this theorem is interesting is that many explicit formulas were developed for the coefficients $Q^{(k,\ell,0,m)}_{n,132}(x)\big|_{x^0}$ and $Q^{(k,\ell,0,m)}_{n,132}(x)\big|_{x^1}$ by Kiteav, Remmel, and Tiefenbruck.
- We also prove the following theorems giving formula about the highest power of x in all generating functions $Q^{(a,b,c,d)}_{n,132}(x)$.
  Theorem 5. If $n \ge k + \ell + 1$, then
$$Q^{(0,k,0,\ell)}_{n,123}(x)\big|_{x^{n-k-\ell}} = Q^{(0,k,0,\ell)}_{n,132}(x)\big|_{x^{n-k-\ell}} = C_k C_{n-k-\ell} C_\ell,$$
$$Q^{(\emptyset,k,0,\ell)}_{n,123}(x)\big|_{x^{n-k-\ell}} = Q^{(\emptyset,k,0,\ell)}_{n,132}(x)\big|_{x^{n-k-\ell}} = C_k C_\ell,$$
$$Q^{(0,k,\emptyset,\ell)}_{n,123}(x)\big|_{x^{n-k-\ell}} = Q^{(0,k,\emptyset,\ell)}_{n,132}(x)\big|_{x^{n-k-\ell}} = C_k C_\ell, \text{ and}$$
$$Q^{(k,\ell,0,0)}_{n,123}(x)\big|_{x^{n-k-\ell}} = Q^{(k,\ell,0,0)}_{n,132}(x)\big|_{x^{n-k-\ell}} = \tfrac{k+1}{k+\ell+1}\tbinom{k+2\ell}{\ell}.$$
  Theorem 6. For $n \ge k + \ell + m + 1$ and $k > 0$,
$$Q^{(k,\ell,0,m)}_{n,123}(x)\big|_{x^{n-k-\ell-m}} = Q^{(k,\ell,0,m)}_{n,132}(x)\big|_{x^{n-k-\ell-m}} = \tfrac{(k+1)^2}{(k+\ell+1)(k+m+1)}\tbinom{k+2\ell}{\ell}\tbinom{k+2m}{m}.$$
- The only generating functions $Q^{(a,b,c,d)}_{123}(t, x)$ which we can not compute via **Theorem 1** are generating functions of the form $\mathbf{Q^{(0,b,0,d)}_{123}(t, x)}$.
- In the case $\mathbf{Q^{(0,k,0,0)}_{123}(t, x)}$, we solve the generating function by separately track peaks and non-peaks.
- We define
$$Q^{(0,\binom{k_1}{k_2},0,0)}_{123}(t, x_0, x_1) = \sum_{n=0}^{\infty} t^n Q^{(0,\binom{k_1}{k_2},0,0)}_{n,123}(x_0, x_1),$$
  where $Q^{(0,\binom{k_1}{k_2},0,0)}_{n,123}(x_0, x_1)$
$$= \sum_{\sigma \in \mathcal{S}_n(123)} x_0^{\text{MMP}(0,k_1,0,0)\text{-mch of peaks}} x_1^{\text{MMP}(0,k_2,0,0)\text{-mch of non-peaks}}.$$
- We can show that $Q^{(0,\binom{k_1}{k_2},0,0)}_{n,123}(x_0, x_1)$ satisfies simple recursions which lead to recursive formulas to compute $Q^{(0,\binom{k_1}{k_2},0,0)}_{123}(t, x_0, x_1)$.
- For example, we prove that
  Theorem 7. For all $k_1, k_2 > 0$, we have
$$Q^{(0,\binom{k_1}{0},0,0)}_{123}(t, x_0, x_1)$$
$$= \frac{1}{1 - tx_1 Q^{(0,\binom{0}{0},0,0)}_{123}(t, x_0, x_1)} \Bigg(1 + tQ^{(0,\binom{k_1-1}{0},0,0)}_{123}(t, x_0, x_1)$$
$$+ tx_1 \sum_{i=2}^{k_1-1} t^{i-1} Q^{(0,\binom{0}{0},0,0)}_{i-1,123}(1, x_1) Q^{(0,\binom{k_1-i}{0},0,0)}_{123}(t, x_0, x_1)$$
$$- tx_1 Q^{(0,\binom{0}{0},0,0)}_{123}(t, x_0, x_1) \sum_{i=0}^{k_1-2} t^i Q^{(0,\binom{0}{0},0,0)}_{i-1,123}(1, x_1)\Bigg).$$
- For generating functions $\mathbf{Q^{(0,k,0,\ell)}_{123}(x, t)}$, we divide the graph of a permutation into several regions and have the following theorem to enumerate the coefficients $Q^{(0,k,0,\ell)}_{n,123}(x)|_{x^s}$.
  Theorem 8. For any 123-avoiding permutation $\sigma = \sigma_1 \dots \sigma_n$, σ_j matches MMP$(0, k, 0, \ell)$ in σ if and only if, in the graph $G(\sigma)$ of σ, (j, σ_j) does not lie in the top k rows or the bottom ℓ rows and it does not lie in the left-most k columns or the right-most ℓ columns. Thus
$$\text{mmp}^{(0,k,0,\ell)}(\sigma) = \Big|\{j | k < j \le n - \ell \text{ and } k < \sigma_j \le n - \ell\}\Big|.$$
- Using **Theorem 8**, we can calculate $Q^{(0,k,0,\ell)}_{123}(t, x)$ for k, l not too big. For example,
  Theorem 9. For $n \ge 4$, $Q^{(0,1,0,1)}_{123}(t, x)\big|_{t^n x^k} = 0$ unless $k \in \{n-4, n-3, n-2\}$ and
$$Q^{(0,1,0,1)}_{123}(t, x)\big|_{t^n x^{n-4}} = C_n - 2C_{n-1} + C_{n-2} - 2,$$
$$Q^{(0,1,0,1)}_{123}(t, x)\big|_{t^n x^{n-3}} = 2C_{n-1} - 2C_{n-2} + 2, \text{ and}$$
$$Q^{(0,1,0,1)}_{123}(t, x)\big|_{t^n x^{n-2}} = C_{n-2}$$
  where C_n denote the n^{th} Catalan number.

Open Problems

We do not have recursion for $Q^{(0,k,0,\ell)}_{123}(x, t)$.

Conjecture. For all $k \ge 1$, we have
$$Q^{(0,k,0,0)}_{132}(t, x) = Q^{(1,k-1,0,0)}_{132}(t, x).$$

Dun Qiu and Jeffery Remmel
duqiu@ucsd.edu, remmel@math.ucsd.edu
University of California San Diego, United States