The Möbius function of increasing oscillations

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Outline

Burstein, Jelínek, Jelínková and Steingrímsson [1] found two recurrences for the Möbius function for decomposable permutations.

We build on their work to find a recurrence where the upper and lower permutations are indecomposable.

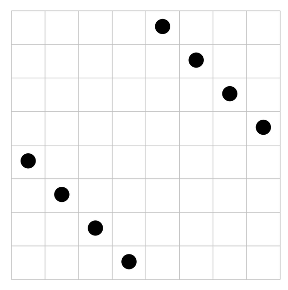
We use this to find an $O(n^2)$ method for determining the Möbius function for increasing oscillations where the lower bound is sum indecomposable.

Preliminaries

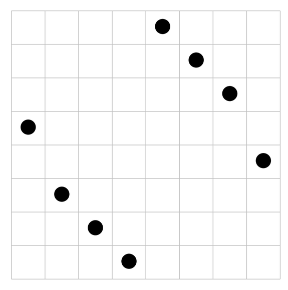
The Möbius function is defined on an interval $[\sigma, \pi]$ of a poset as 0 if $\sigma \not\leq \pi$, 1 if $\sigma = \pi$, and $-\sum_{\sigma \leq \lambda < \pi} \mu[\sigma, \lambda]$ otherwise.

The *direct interleave* of two permutations α and β is formed by taking the direct sum $\alpha \oplus \beta$, and then exchanging the value of the largest point from α with the value of the smallest point from β. We can also view this as increasing the largest point from α by 1, and simultaneously decreasing the smallest point from β by 1. We write an interleave as $\alpha \oslash \beta$.

If α is a permutation, and r a positive integer, then $r.\,\alpha$ is $\alpha \oplus \ldots \oplus \alpha$, with r occurrences of α. Similarly, α^r is $\alpha \oslash \ldots \oslash \alpha$, with r occurrences of α.



$4321 \oplus 4321$



$4321 \oslash 4321$

General results

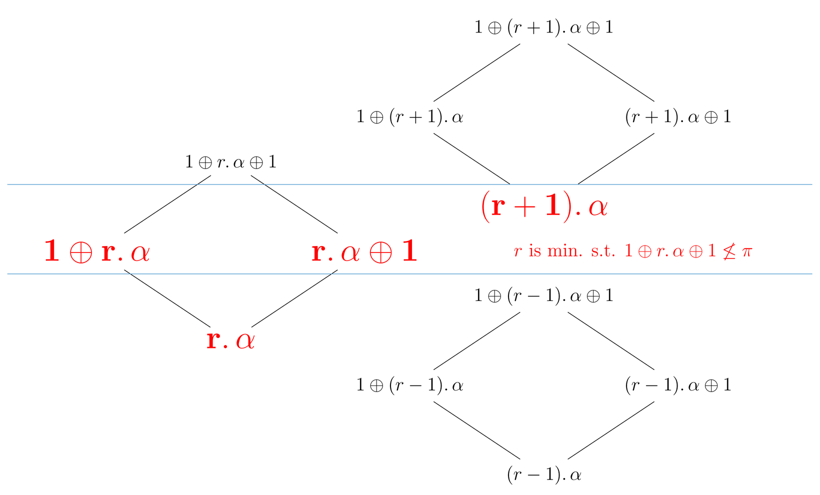
Burstein, Jelínek, Jelínková and Steingrímsson [1] found two recurrences for the Möbius function for decomposable permutations. A simple consequence of their work is that if σ is indecomposable, then for any π, the non-zero contributors to the Möbius function are either 1, or 12, or have one of the forms

$$r.\,\alpha, \qquad 1 \oplus r.\,\alpha, \qquad r.\,\alpha \oplus 1 \qquad \text{or} \qquad 1 \oplus r.\,\alpha \oplus 1$$

where α is indecomposable.

We can also show that for any α there are at most two values of r where the contribution can be non-zero.

This gives us a $\{0, \pm 1\}$ weighting function $W(\sigma, \alpha, \pi)$ which gives the contribution to the Möbius function for any indecomposable α.



Theorem

If σ is a sum indecomposable permutation, and π has length at least 4, then

$$\mu[\sigma, \pi] = -\sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

This is an improvement on the standard sum since the number of elements in $\mathfrak{C}_{\sigma,\pi}$ is less than the number of elements in the poset interval $[\sigma, \pi)$.

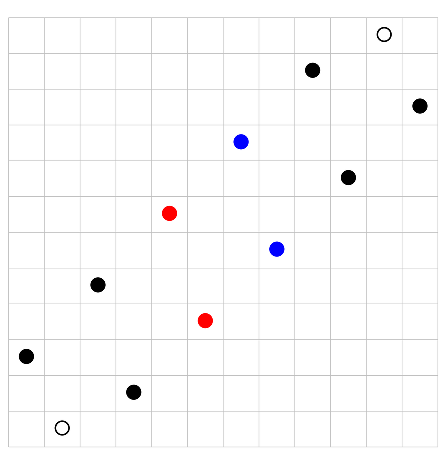
Increasing oscillations

An *increasing oscillation* is a simple permutation order-isomorphic to a sub-sequence of

$$4, 1, 6, 3, 8, 5, \ldots, 2k+2, 2k-1, \ldots$$

.

A template for all increasing oscillations is shown to the right. The hollow points are optional. The length can be changed by inserting or deleting pairs of points as highlighted by the coloured points.



An increasing oscillation with n elements that starts with a descent is called W_n. If it starts with an ascent, it is called M_n.

[1] A. Burstein, V. Jelínek, E. Jelínková and E. Steingrímsson. The Möbius function of separable and decomposable permutations. *Journal of Combinatorial Theory, Series A*, 118(8):2346-2364, 2011.

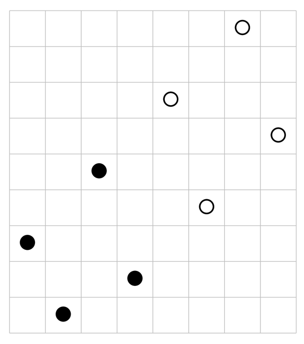
Indecomposable permutations in an increasing oscillation

Building on the general results, we can show that every indecomposable permutation contained in an increasing oscillation can be written as

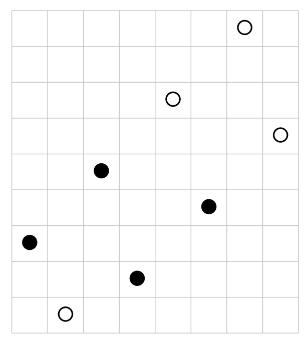
$$(21)^k, \qquad 1 \oslash (21)^k, \qquad (21)^k \oslash 1, \qquad \text{or} \qquad 1 \oslash (21)^k \oslash 1$$

For each shape of indecomposable permutation, and each shape of increasing oscillation, we can find an inequality that relates the size of the indecomposable permutation, the number of copies of the indecomposable permutation and the size of the increasing oscillation. Here are some examples minimizing the unused elements at the start of an increasing oscillation.

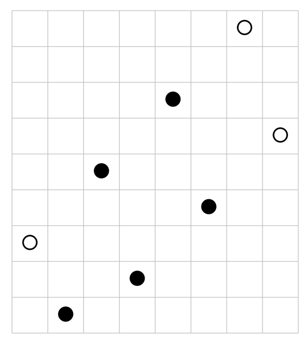
$(21)^2$



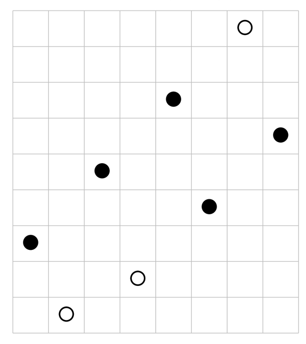
$1 \oslash (21) \oslash 1$



$1 \oplus (21)^2$



$1 \oplus 1 \oslash (21) \oslash 1$



This leads to the following inequalities; which relate σ or π, the shape of α, the size of α as given by k, and the value of r. In these tables α^{k^+} indicates that $k \geq 2$.

σ	Shape of α	Inequality
$W_{2n-1}, M_{2n}, M_{2n-1}$	(21)	False
W_{2n-1}	$(21)^k \oslash 1$	$k \geq n-1$
M_{2n-1}	$1 \oslash (21)^k$	$k \geq n-1$
$W_{2n-1}, M_{2n}, M_{2n-1}$	$1 \oslash (21)^k \oslash 1$	$k \geq n-1$
W_{2n-1}	$1 \oslash (21)^k$	$k \geq n+1$
M_{2n}	$(21)^{k^+}$	$k \geq n+1$
M_{2n-1}	$(21)^k \oslash 1$	$k \geq n+1$
All other cases		$k \geq n$

π	Shape of α	Inequality
W_{2n}, M_{2n}	(21)	$3r - 1 \leq 2n$
W_{2n-1}, M_{2n-1}	(21)	$3r \leq 2n$
W_{2n}	$(21)^{k^+}$	$2kr + 2r - 2 \leq 2n$
W_{2n-1}, M_{2n-1}	$1 \oslash (21)^k$	$2kr + 2r + 2 \leq 2n$
M_{2n}	$1 \oslash (21)^k \oslash 1$	$2kr + 4r - 2 \leq 2n$
$W_{2n}, W_{2n-1}, M_{2n-1}$	$1 \oslash (21)^k \oslash 1$	$2kr + 4r \leq 2n$
All other cases		$2kr + 2r \leq 2n$

$\mu[\sigma, W_n]$ and $\mu[\sigma, M_n]$, σ indecomposable

Theorem

If σ is a sum indecomposable permutation, and π is an increasing oscillation with length at least 4, then

$$\mu[\sigma, \pi] = \sum_{\alpha \in \mathfrak{S}} \sum_{v=\mathrm{MinK}(\sigma,\alpha)}^{\mathrm{MaxK}(\alpha,\pi)} \mu[\sigma, \alpha_v] W(\sigma, \alpha_v, \pi)$$

where the first sum is over the possible shapes of a sum indecomposable permutation contained in an increasing oscillation, so $\mathfrak{S} = \{(21), (21)^{k^+}, 1 \oslash (21)^k, (21)^k \oslash 1, 1 \oslash (21)^k \oslash 1\}$.

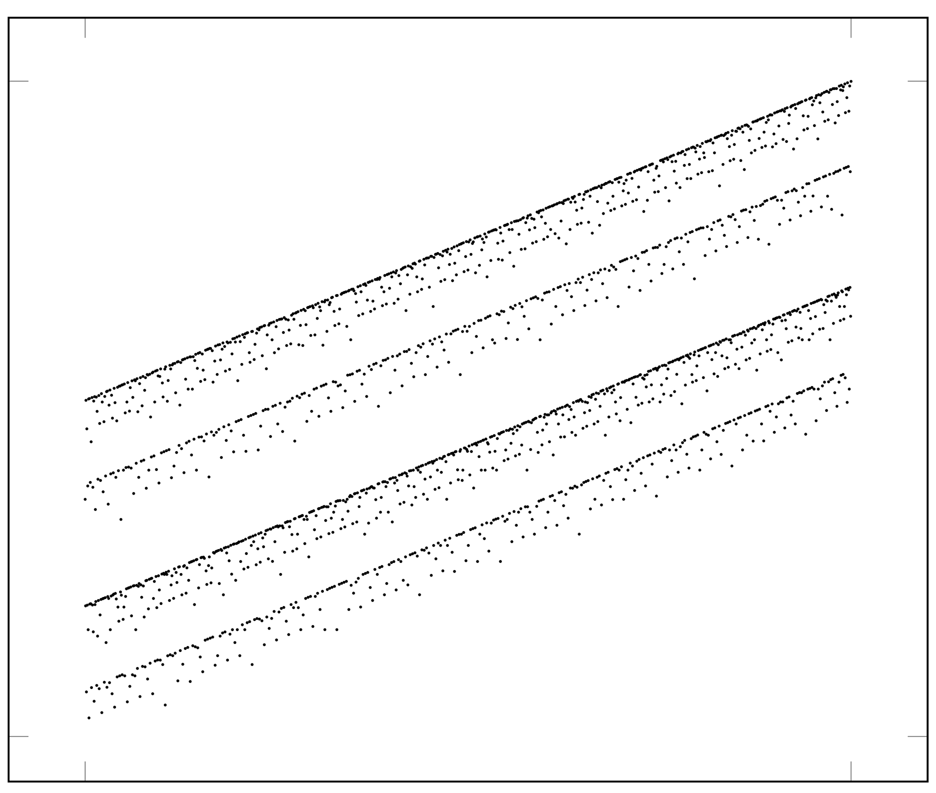
The weighting function here is slightly simpler than the general case.

Using the inequalities gives us an $O(n^2)$ algorithm for determining the Möbius function of an increasing oscillation where the lower bound is indecomposable.

As an example, we were able to find $\mu[1, W_{1000000}] = -222222444441$.

Asymptotic behaviour

Here is a log-log plot of the absolute value of W_{2n} for $n = 8000 \ldots 10000$



Conjecture

Based on the numerical evidence found so far, we conjecture that

- W_{2n} is in the interval $[-n^2, -\Phi n^2]$
- W_{2n-1} is in the interval $[\Phi(n^2 - n), n^2 - n]$

where $\Phi = \frac{\sqrt{5}-1}{2} \approx 0.618$

We have no explanation for this pattern.