Stirling Numbers & Euler's Finite Difference Theorem

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Abstract

We generalize Euler's Finite Difference Theorem [5] to produce a polynomial identity involving *Stirling numbers of the second kind*, denoted by $\left\{ {n \atop k} \right\}$. Applications of this formula are given.

Introduction

Stepping back about 10 years, Dence [2] uses the Laplace transform of $\sin^n(t)$ to generate the following two identities for positive integers n, k:

$$\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{j} (n-2j)^k = \begin{cases} (2)^{n-1} n!, & k = n, \text{ and} \\ 0, & 0 \le k < n \end{cases}, \tag{1}$$

and

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{n-2j} \binom{n}{j} = (-1)^{\lfloor n/2 \rfloor} \frac{2^{n-1} n!}{(1 \cdot 3 \cdot 5 \cdot \cdots \cdot n)^2}, \text{ if } n \text{ is odd.} \tag{2}$$

Two years later, Katsuura [3] proves a generalization of Equation (1), namely, for any $x, y \in \mathbb{C}$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (y + jx)^k = \begin{cases} (-x)^n n!, & k = n, \text{ and} \\ 0, & 0 \le k < n. \end{cases} \tag{3}$$

Katsuura remarks that it is "curious" that no y term appears on the righthand side of this equation a fact which is explained in 2014 by proving Equation (3) using involutions on colored words [4]. While this combinatorial proof addresses only the case where $0 \le k \le n$, that stipulation on the relative order of k, n is artificial. Indeed, using the same involution, we get the following result, of which Equations (1) and (3) are corollaries.

Theorem 1. *For nonnegative integers n, k and for any $x, y \in \mathbb{C}$,*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (jx + y)^k = (-1)^n n! \sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop n} \right\} x^i y^{k-i}. \tag{4}$$

In this way, viewed purely as a polynomial identity, we see from the righthand side that, due to the presence of the $\left\{ {i \atop n} \right\}$, positive powers of y exist only in the case where $k > n$, i.e., the aforementioned curiosity is explained. Further, there is nothing stopping us from replacing $x, y \in \mathbb{C}$ with some other indeterminate variables X, Y.

Stepping back for a moment, and realizing that very few simple formulae involving Stirling numbers are *new*, we see that Equation (3) is really a direct application of the following theorem.

Theorem 2 (Euler's Finite Difference Theorem [5])**.** *Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ be a complex polynomial and let n be a nonnegative integer. Then*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} f(j) = \begin{cases} (-1)^n a_n n!, & k = n, \text{ and} \\ 0, & 0 \le k < n. \end{cases}$$

Our goal is to generalize Theorem 2 and then use that generalization for our own combinatorial purposes. To that end, we start with the following well-known formula involving Stirling numbers of the second kind:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k = (-1)^n n! \left\{ {k \atop n} \right\}.$$

Letting $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we multiply both sides of the preceding equation by a_k and sum over all $k \ge 0$ to obtain our desired generalization of Theorem 2:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} g(j) = (-1)^n n! \sum_{k \ge 0} a_k \left\{ {k \atop n} \right\}. \tag{5}$$

Results

Equation (5) is useful in directly generating combinatorial identities.

1. Let
$$g(j) = (jx + y)^k = \sum_{i=0}^{k} \binom{k}{i} (jx)^i y^{k-i} = \sum_{i=0}^{k} \left(\binom{k}{i} x^i y^{k-i} \right) j^i,$$
giving us Theorem 1, which was the motivating identity.

2. Let $g(j) = 1$, giving that $a_0 = 1$ and $a_k = 0$ for $k \ge 1$. This yields the well-known alternating binomial sum identity
$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0.$$

3. Let $g(j) = \cos(\pi j) = \sum_{i \ge 0} \frac{(-1)^i (\pi j)^{2i}}{(2i)!}$. Then $\sum_{j=0}^{n} (-1)^j \binom{n}{j} g(j) = \sum_{j=0}^{n} \binom{n}{j} = 2^n$, so that
$$2^n = (-1)^n n! \sum_{k \ge 0} \frac{(-1)^k \pi^{2k}}{(2k)!} \left\{ {2k \atop n} \right\}.$$

4. Let $g(j) = \binom{j}{m} = \frac{(j)\downarrow_m}{m!} = \frac{1}{m!} \cdot j(j-1)\cdots(j-(m-1))$ for $0 \le m \le n$. Then
$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{m} = \frac{(-1)^n n!}{m!} \sum_{k \ge 0} \left[{m \atop k} \right] \left\{ {k \atop n} \right\},$$
where $\left[{m \atop k} \right]$ denotes the *signed Stirling number of the first kind.* Since the lefthand side is equal to 0 for $0 \le m < n$ and $(-1)^n$ when $n = m$, we quickly obtain the well-known orthogonality relation between Stirling numbers of the first and second kind.
Similarly, letting $g(j) = (j)\uparrow_m = j(j+1)\cdots(j+(m-1))$, we obtain
$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (j)\uparrow_m = (-1)^n n! \sum_{k \ge 0} (-1)^{m-k} \left[{m \atop k} \right] \left\{ {k \atop n} \right\},$$
which can be rewritten to yield a formula for the Lah numbers, $\mathcal{L}(m, n)$:
$$\mathcal{L}(m, n) = \frac{(-1)^n}{n!} \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} (j)\uparrow_m .$$

5. Broder [1] denotes the r-Stirling numbers of the first and second kind as $\left[{n \atop k} \right]_r$ and $\left\{ {n \atop k} \right\}_r$, respectively. $\left\{ {n \atop k} \right\}_r$ is defined to be the number of set partitions of $\{1, 2, \ldots, n\}$ into k nonempty, unordered parts such that $1, 2, \ldots, r$ are in distinct parts, and thus, are all minimal elements of their parts. The numbers $\left[{n \atop k} \right]_r$ are defined similarly for permutations of $\{1, 2, \ldots, n\}$ into k disjoint cycles. Rewriting Equation (32) from [1] and then applying Theorem 1 yields
$$(-1)^n n! \left\{ {k+r \atop n+r} \right\}_r = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (j+r)^k. \tag{6}$$
Multiplying both sides of the Equation (6) by $z^k/k!$ and summing over $k \ge 0$ gives us the following exponential generating function of the r-Stirling numbers:
$$\sum_{k \ge 0} \left\{ {k+r \atop n+r} \right\}_r \frac{z^k}{k!} = \frac{e^{zr}}{n!} (e^z - 1)^n.$$
Unlike the original proof, this does not require any knowledge of the generating functions of Stirling numbers themselves.

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References

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