Peaks, Descents, and Pattern Avoidance

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Let $\operatorname{des}(\pi)$ be the number of descents and $\operatorname{pk}(\pi)$ the number of peaks of a permutation π. In [1], Brändén proved that for any subset $\Pi \subseteq \mathfrak{S}_n$ invariant under a group action called the *modified Foata–Strehl action* (abbreviated as *MFS*), the descent polynomial $A(\Pi; t) := \sum_{\pi\in\Pi} t^{\operatorname{des}(\pi)}$ is γ-positive and is related to the peak polynomial $P(\Pi; t) := \sum_{\pi\in\Pi} t^{\operatorname{pk}(\pi)}$ by the formula

$$A(\Pi; t) = \left(\frac{1+t}{2}\right)^{n-1} P\left(\Pi; \frac{4t}{(1+t)^2}\right).$$

By taking $\Pi = \mathfrak{S}_n$, this yields well-known results of Foata–Schützenberger [2] and Stembridge [4] on Eulerian polynomials.

In this talk, we produce a refinement of Brändén's formula: For any $\Pi \subseteq \mathfrak{S}_n$ invariant under the modified Foata–Strehl action, the descent polynomial $A(\Pi; t)$ and the polynomial $P(\Pi; y, t) := \sum_{\pi\in\Pi} y^{\operatorname{pk}(\pi)} t^{\operatorname{des}(\pi)}$ encoding the joint distribution of the peak number and descent number over Π satisfy the relation

$$A(\Pi; t) = \left(\frac{1+yt}{1+y}\right)^{n-1} P\left(\Pi; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right).$$

Setting $y = 1$ recovers Brändén's formula, and taking $\Pi = \mathfrak{S}_n$ yields an analogous result on Eulerian polynomials which can also be proven using noncommutative symmetric functions (see [5]).

Several subsets $\Pi \subseteq \mathfrak{S}_n$ known to be invariant under the modified Foata–Strehl action can be characterized in terms of pattern avoidance. For example, Brändén [1] showed that the pattern class $\operatorname{Av}_n(231)$ is MFS-invariant, and Kim and Lin [3] showed that $\operatorname{Av}_n(3142, 1342)$ is also MFS-invariant. Thus, we pose the question: Can we characterize all pattern classes invariant under the modified Foata–Strehl action? We conclude the talk with some preliminary results in this direction, which is joint work with Richard Zhou (Lexington High School).

References

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