Packing Densities: An Update

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We summarize what we know about packing densities of permutations of size 4.

If S is a permutation ("pattern") of size k and P is any permutation, then $p(S, P)$ is the fraction of k-subsequences of P that are order-isomorphic to S. If we maximize over permutations P of a given size n and then let $n \to \infty$, we get the *packing density* $p(S)$ of the pattern itself. For example, it is known that $p(132) = 2\sqrt{3} - 3$.

Packing densities can also be defined for *permutons* [HKM, PS]. A permuton μ is a probability measure on the square $[0, 1]^2$ with uniform marginals. Now $p(S, \mu)$ is the probability that k points chosen independently from μ are ordered like the graph of S. Maximizing over μ gives the same $p(S)$ as above.

Here are the known bounds for $p(S)$ when $S \in \mathcal{G}_4$. Up to symmetry, these are all of the cases.

<table>
<tr><th>S</th><th>upper bound</th><th>lower bound</th></tr>
<tr><td>1234</td><td colspan="2">1</td></tr>
<tr><td>1243</td><td colspan="2">3/8</td></tr>
<tr><td>2143</td><td colspan="2">3/8</td></tr>
<tr><td>1432</td><td colspan="2">algebraic, ≈ 0.4236</td></tr>
<tr><td>1324</td><td colspan="2">0.244054523929423329586873 3...</td></tr>
<tr><td>1342</td><td>≈ 0.198836597</td><td>≈ 0.198837287</td></tr>
<tr><td>2413</td><td>≈ 0.104724</td><td>≈ 0.104780</td></tr>
</table>

Our tool for finding upper bounds is the technique of flag algebras, as introduced by Razborov in 2007. We are using *Permpack*, the implementation of flag algebras by Jakob Sliačan that is specialized for permutations and that was described at PP2016.

Our main result is the new lower bound for $p(1342)$. It uses a new construction, replacing the elegant but (alas) not optimal construction of Batkeyev. The upper bound is found using flag algebras. Our calculation of the upper bound is inexact, to the extent that we see the closeness of the upper and lower bound as evidence that the new construction may really be optimal.

We also present a more precise value for $p(1324)$, which we can now compute to at least 100 digits using Mathematica. We do not know whether this packing density is an algebraic number. Flag algebras give us an independent upper bound of 0.244054549 for $p(1324)$.

Flag algebras also give the upper bound for $p(2413)$, and its closeness to the lower bound is evidence that the complicated construction in [PS] (which gives the lower bound) may be in the right direction.

Beyond addressing packing densities, the technique can reveal surprising relationships among the packing statistics for different patterns. In this talk, we hope to describe the flag-algebra technique using small examples. For a start, we will prove using explicit human-sized calculations the well-known bound

$$p(132) \leq 2\sqrt{3} - 2.$$

We will also give an independent proof the known result that

$$p(123, \mu) + p(321, \mu) \geq 1/4$$

and the bound

$$p(213, \mu) + p(312, \mu) \leq 1/2.$$

The last two results hold for permutons, and also for permutations in the limit of large n.

References

[HKM] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Rth, and M. R. Sampaio M. R., Limits of permutation sequences, *Journal of Combinatorial Theory, Series B,* 103(1):93-113, 2013.

[PS] C. B. Presutti and W. Stromquist, Packing rates of measures and a conjecture for the packing density of 2413, *Permutation patterns* 376:287316, 2010.