Interval minors of binary matrices *

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In the study of Stanley–Wilf limits, Fox [1] introduced a new relation of pattern avoidance for binary matrices – interval minors.

For a binary matrix M, the *contraction* of two adjacent rows is an operation that replaces the two rows by a single row that has one-entries in positions where at least one of the previous rows had a one-entry. The same can be applied to any two adjacent columns. We say that a binary matrix P (pattern) is an *interval minor* of a binary matrix M and denote it by $P \preceq M$, if P can be obtained from M by applying a sequence of row and column contractions and one-entry deletions (changes of one-entries to zero-entries). We also assume that all contractions are done before deletions of one-entries and so every sequence of operations creates a natural partitioning of the matrix M. If P is not an interval minor of M, we say that M *avoids* P, denote it by $P \not\preceq M$ and by $Av_{\preceq}(P)$ we denote the class of all matrices avoiding P. In Figure 1, we see an example of a pattern P, a matrix M_1 such that $P \preceq M_1$ (with one possible partitioning) and a matrix M_2 avoiding P.

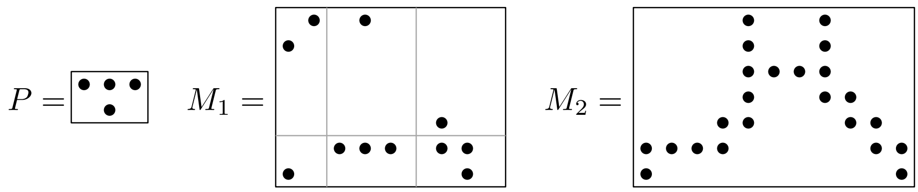

Figure 1: A matrix M_1 containing matrix P as an interval minor and a matrix M_2 avoiding P.

To every permutation π, we may associate its permutation matrix M_π. We may then observe that a permutation π contains a permutation σ if and only if M_σ is an interval minor of M_π. Thus, the notion of interval minors can be seen as a generalization of classical Wilf containment to arbitrary binary matrices.

*This is joint work with Vít Jelínek.

We will present the characterizations of P-avoiding binary matrices for several small patterns P. These characterizations will serve us as a motivation for the study of the properties of general classes of interval minor avoiding matrices.

We say that a binary matrix M is *P-critical* for some matrix P if $P \not\preceq M$ and P is an interval minor of every matrix created from M by changing a zero-entry to one-entry. In Figure 1, the matrix M_2 is P-critical.

A *flip* in a binary string $b = b_1 b_2 \dots b_n$ is a pair of adjacent entries (b_i, b_{i+1}) such that $b_i \neq b_{i+1}$. Considering each line (row or column) of a matrix M to be a binary string, the *line complexity* is the number of flips contained in it. We define the *complexity of a matrix* as the maximum of complexities of its lines. We say that the *complexity of a class $Av_{\preceq}(P)$* is *bounded* if there is an integer k such that each P-critical matrix has its complexity bounded by k. Otherwise, the complexity of the class is *unbounded.* For example, consider again the pattern P and matrix M_2 from Figure 1. We see that the complexity of the matrix M_2 is 4. Moreover, our characterization implies that the complexity of every P-critical matrix is bounded by 4; thus, the complexity of $Av_{\preceq}(P)$ is bounded.

As our main result, we obtain the following characterization of the patterns P for which the P-avoiding matrices have bounded complexity.

Theorem 1. *Let P be a binary matrix. The complexity of $Av_{\preceq}(P)$ is bounded if and only if P avoids the following four matrices P_1, P_2, P_3 and P_4 as interval minors:*

$$P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \; P_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix} \; P_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \; P_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}.$$

References

[1] Jacob Fox, Stanley-Wilf limits are typically exponential. *arXiv:1310.8378*, 2013.