Positional Marked Patterns in Permutations

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Let S_n denote the set of permutations of $1, \ldots, n$. A *positional marked pattern* is pair (τ, s) such that $\tau = \tau_1 \ldots \tau_j \in S_j$ and $1 \leq s \leq j$. For any sequence $\alpha = \alpha_1 \ldots \alpha_k$ of pairwise distinct positive integers, we let $\text{red}(\alpha)$ be the permutation in S_k that results by replacing the i-th smallest element in α by i. For example, $\text{red}(79462) = 45231$. Given a permutation $\tau = \tau_1 \ldots \tau_j \in S_j$ and a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that τ *occurs* in σ if there exists $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \ldots \sigma_{i_j}) = \tau$. We way that σ *avoids* τ if σ has no occurrences of τ. We let $S_n(\tau)$ denote the number of $\sigma \in S_n$ such that σ avoids τ. Then we say that $\alpha \in S_j$ is Wilf-equivalent to $\beta \in S_j$ if and only if $S_n(\alpha) = S_n(\beta)$ for all $n \geq 1$.

Given a positional marked pattern (τ, s) and a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches (τ, s) if there exists $1 \leq i_1 < \cdots < i_{s-1} < i < i_{s+1} \ldots i_j \leq n$ such that $\text{red}(\sigma_{i_1} \ldots \sigma_{i_{s-1}} \sigma_i \sigma_{i_{s+1}} \ldots \sigma_{i_j}) = \tau$. We let $pmp_{(\tau,s)}(\sigma)$ denote the number of i such that σ_i matches (τ, s) in σ. In the language of classical patterns, σ_i matches (τ, s) if there is an occurrence of τ in σ such that σ_i plays the role of τ_s in that occurrence. Note that it is possible that σ_i is part of many occurrences of τ such that σ_i plays the role τ_s in such an occurrence. For example suppose that $(\tau, s) = (123, 1)$ and $\sigma = 145236$. There are 8 occurrences of 123 in σ, namely,

$$145, 146, 156, 123, 126, 136, 456, 236.$$

Then one can see that 1 plays the role of 1 in 5 occurrences of 123 in σ, 4 plays the role of 1 in just one occurrence of 123 in σ, and 2 plays the role of 1 in just one occurrence of 123. We do not care how many times 1 can play the role of 1 in an occurrence of 123 in σ. That is, as long as there at one occurrence, it will contribute 1 to $pmp_{(123,1)}(\sigma)$. In this case, we see that $pmp_{(123,1)}(\sigma) = 3$. Note, however, that 2, 3, 4, and 5 can play the role of 2 on an occurrence of 123 in σ so that $pmp_{(123,2)}(\sigma) = 4$. Finally 3, 5 and 6 can play the role of 3 in an occurrence of 123 in σ so that $pmp_{(123,3)}(\sigma) = 3$.

The goal of this paper is to study polynomials of the form

$$P_{n,(\tau,s)}(x) = \sum_{\sigma \in S_n} x^{pmp_{(\tau,s)}(\sigma)}. \tag{1}$$

Given $\alpha = \alpha_1 \ldots \alpha_j$ and $\beta = \beta_1 \ldots \beta_j$ in S_j and $1 \leq s, t \leq j$, we say that (α, s) is **positional Wilf-equivalent** to (β, t), written $(\alpha, s) \backsim_p (\beta, t)$, if and only if, for all $n \geq 1$,

$$P_{n,(\alpha,s)}(x) = P_{n,(\beta,t)}(x). \tag{2}$$

We note that for any s, $P_{n,(\tau,s)}(0)$ is just the number of permutations $\sigma \in S_n$ such that σ has no occurrences of τ, i.e., the number of permutations of σ that avoid τ. It follows that if (α, s) is positional Wilf-equivalent to (β, t), then α and β are Wilf-equivalent. Our second goal is to find the positionally Wilf-equivalence classes for small n. We note that positional Wilf-equivalence is preserved by the group of symmetries of the square in the obvious way. For example, suppose

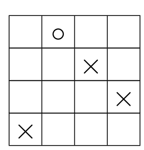
$\alpha \in S_j$ and $1 \leq s \leq j$. If $\sigma = \sigma_1 \ldots \sigma_n$ and we let $\sigma^r = \sigma_n \ldots \sigma_1$ be the reverse of σ and $\sigma^c = (n+1-\sigma_1) \ldots (n+1-\sigma_n)$, then clearly

$$\begin{aligned} (\alpha, s) &\backsim_p (\alpha^r, j+1-s), \\ (\alpha, s) &\backsim_p (\alpha^c, s), \text{ and} \\ (\alpha, s) &\backsim_p (\alpha^{-1}, \alpha_s). \end{aligned}$$

We shall see that positional Wilf-equivalence is a refinement of Wilf-Equivalence. For example, we know that all permutations of length 3 are Wilf-equivalent, but we shall show that there exactly two positional Wilf-equivalence classes among the permutations of length 3.

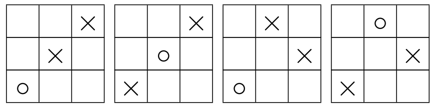
We note that positional marked patterns can be viewed as a kind of refinement of quadrant marked mesh patterns. The notion of mesh patterns was first introduced by Brändén and Claesson [**?**]. Many authors have further studied this notion. In particular, the notion of marked mesh pattern was introduced by Úlfarsson [**?**], and the study of the distributions of quadrant marked mesh patterns in permutations was initiated by Kitaev and Remmel in [**?**]. Here we say that an element σ_i in a permutation $\sigma_1 \ldots \sigma_n$ matches the quadrant marked mesh pattern $MMP(a,b,c,d)$ where a,b,c,d are non-negative integers if σ_i has at least a to its right which are larger than it, at least b elements to its left which are larger than it, at least c elements to its left which are smaller than it, and at least d elements to its right which are smaller than it. If we think that graph of σ, i.e, $\{(i,\sigma_i) : i = 1,\ldots,n\}$, with coordinate system centered at the point (i,σ_i), then σ_i matches $MMP(a,b,c,d)$ if and only if there are at least a elements in the first quadrant, at least b elements in its second quadrant, at least c elements in its third quadrant, and at least d elements in its fourth quadrant. For σ_i to match the positional marked pattern (τ, s), we would be specifying that not only do certain elements exist in each quadrant, but also the relative position of all these elements with respect to each other.

For small permutation, we shall often denote the positional marked pattern (τ, s) by simply underlining the τ_s in τ. For example, we shall denote $(123, 1)$ by $\underline{1}23$. We let S_n^* denote the set of permutations σ in S_n where exactly one element is underlined. Then positional marked mesh patterns (τ, s) where $\tau \in S_n$ and $1 \leq s \leq n$ can be identified with elements of S_n^*. We will also associate a positional marked pattern with a permutation matrix-like diagram. Given any $\tau \in S_k^*$, the diagram associated to τ is a k by k array with the filling such that if $i \neq s$, then we put an $\times$ in cell (i,σ_i) and if $i = s$, then $\circ$ in cell (i,σ_i). For example, if $\tau = 1\underline{4}32$, then the corresponding diagram is



It is then easy to see that for $\alpha, \beta \in S_n$ and $1 \leq s,t \leq n$, (α, s) is positional Wilf-equivalent to (β, t) if the diagram of (α, s) can be transformed into the diagram of (β, t) via a series of rotations and reflections of the square.

For example, consider the positional marked patterns of length 3. There are $3! \cdot 3 = 18$ positional marked patterns of length 3. However, with symmetry, there are at most 4 equivalence classes, which are represented by $\underline{1}23, 1\underline{2}3, \underline{1}32, 1\underline{3}2$. Their diagrams are shown below:



Our next result implies that there exactly 2 positional Wilf-equivalence classes of length 3.

Theorem 1. $\underline{1}23$ *and* $\underline{1}32$ *are positional Wilf-equivalent.* $1\underline{2}3$ *and* $1\underline{3}2$ *are positional Wilf-equivalent. However,* $\underline{1}23$ *is not positional Wilf-equivalent to* $1\underline{2}3$.

For the positional patterns of length 4, there are 16 positional patterns up to the symmetries of the square. Computational evidence suggests that in fact there are only 10 positional Wilf-equivalences classes. By directly computing the polynomials $P_{n,(\tau,s)}(x)$ where $\tau \in S_4$ and $1 \leq s \leq 4$, we know that there are at least 10 positional Wilf-equivalence classes. While we have not been able to complete the classification of such positional Wilf-equivalence classes, we have proved the following results.

Theorem 2. $1\underline{2}34$*,* $1\underline{2}43$*,* $\underline{2}134$ *and* $\underline{2}143$ *are equivalent.*

Theorem 3. *Let* $p_1, p_2, \ldots, p_k$ *be a rearrangement of* $3, 4, \ldots, k+2$*, then* $1\underline{2}p_1p_2\ldots p_k$ *and* $\underline{2}1p_1p_2\ldots p_k$*. are positional Wilf-equivalent.*

We study the coefficients of $P_{n,(\tau,s)}(x)$. Given any polynomial $f(x)$, we let $f(x)|_{x^k}$ denote the coefficient of x^k in $f(x)$. For example, our observation above shows that $P_{n,(\tau,s)}(x)|_{x^0} = |S_n(\tau)|$. We have proved a number of general results about the coefficient $P_{n,(\tau,s)}(x)|_{x^k}$. Below we list a two examples.

Theorem 4. $P_{n,\underline{1}32}(x)|_x = \sum_{\sigma \in S_n(132)} inv(\sigma)$

Theorem 5. *Given any positive integers* n, k *such that* $n \geq k$*, and any* $\tau \in S_k$ *and* $1 \leq s \leq k$*, the degree of* $P_{n,(\tau,s)}(x)$ *is* $n-k+1$*, and* $P_{n,(\tau,s)}(x)|_{x^{n-k+1}} = (n-k+1)!$.

References

[BC] P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, Electronic J. Combin. **18** (2) (2011), Paper #P5.

[KR] S. Kitaev and J. Remmel, Quadrant marked mesh patterns, Journal of Integer Sequence, **15** Issue 4 (2012), Article 12.4.7, 29 pgs.

[U] Henning A. Úlfarsson, A unification of permutation patterns related to Schubert varieties, DMTCS proc., **AN** (2010), 1057-1068.

[BW] E. Babson and J. West, The Permutations 123p4 . . . pm and 321p4 . . . pm are Wilf-Equivalent, Graphs and Combinatorics, **16:173**, (2000), 373-380