Inversion Sequences and Generating Trees

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1 Introduction

An *inversion sequence* is an integer sequence $(e_1, \ldots, e_n)$ satisfying $0 \leq e_i < i$ for all $i = 1, \ldots, n$. There is a natural bijection $\Theta : S_n \to I_n$ from S_n, the set of permutations of length n, to I_n, the set of inversion sequences of length n. Under this bijection, $e = \Theta(\pi)$ is obtained from a permutation $\pi = \pi_1 \ldots \pi_n \in S_n$ by setting $e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|$. The study of patterns in inversion sequences was introduced in [9] and [13]. In [9] the authors study inversion sequences that avoid words of length 3, while in [13] inversion sequences avoiding permutations of length 3 are considered. For instance, the inversion sequences that avoid the pattern 110 are those with no $i < j < k$ such that $e_i = e_j > e_k$. This study has been carried on in [14], where the notion of pattern avoidance is generalized to a triple of binary relations (ρ_1, ρ_2, ρ_3) and study the set $I_n(\rho_1, \rho_2, \rho_3)$ of those $e \in I_n$ with no $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$, and $e_i \rho_3 e_k$. For example $I_n(=, >, >) = I_n(110)$. In [14] all triples of relations of the set $\{<, >, \leq, \geq, =, \neq, -\}^3$ are studied, where $-$ on a set S is the cartesian product, i.e. $- = S \times S$. Therefore all 343 patterns are considered and partitioned in 98 equivalence classes. In [14] the authors found several enumeration results beyond those in [9, 13], and formulated several conjectures, some of which are studied in this paper.

Generating trees and succession rules will also be important for our work. We give a brief general presentation below. Details can be found for instance in [1, 2, 3, 16]. We also review the classical generating tree for Dyck paths and the classical succession rule for Catalan numbers.

Consider any combinatorial class $\mathcal{C}$, that is to say any set of discrete objects equipped with a notion of size, such that there is a finite number of objects of size n for any integer n. Assume also that $\mathcal{C}$ contains exactly one object of size 1. A *generating tree* for $\mathcal{C}$ is an infinite rooted tree, whose vertices are the objects of $\mathcal{C}$, each appearing exactly once in the tree, and such that objects of size n are at distance n from the root (with the convention that the root is at distance 1 from itself). The children of some object $c \in \mathcal{C}$ are obtained by adding an *atom* (*i.e.* a piece of object that makes its size increase by 1) to c. Of course, since every object should appear only once in the tree, not all additions are possible. We should ensure the unique appearance property by considering only additions that follow some restricted rules. We will call the *growth* of $\mathcal{C}$ the process of adding atoms following these prescribed rules.

Such a growth for Dyck paths is described in [2]. Recall that a Dyck path of semilength n is a lattice path using up U and down D unit steps and running from $(0, 0)$ to $(2n, 0)$ remaining weakly above the x-axis. The atoms are *peaks*, that is to say UD factors. To ensure that all Dyck paths appear exactly once in the generating tree, peaks are inserted in the *last descent* of the path, which is the longest suffix containing only the letter D. More precisely, the children of a Dyck path $w \cdot UD^k$ are $w \cdot U\boldsymbol{UD}D^k, w \cdot UD\boldsymbol{UD}D^{k-1}, \ldots, w \cdot UD^{k-1}\boldsymbol{UD}D$ and $w \cdot UD^k\boldsymbol{UD}$. The first few levels of the generating tree for Dyck paths are shown in Figure 1 (left).

Since the growth of $\mathcal{C}$ uniquely defines the shape of the generating tree, for enumerative purpose we choose to identify a generating tree with its shape without regarding the objects that label its nodes. If such growth is particularly regular, we can represent it via a *succession rule*. A succession

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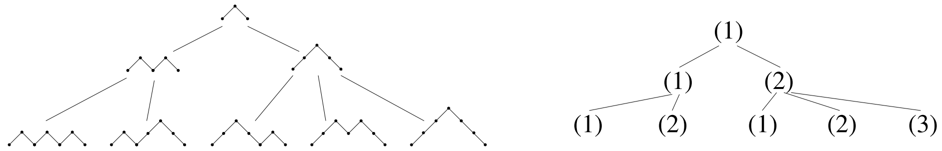

Figure 1: Two ways of looking at the generating tree for Dyck paths: with objects (left) and with labels from the succession rule Ω_{Cat} (right).

rule is made of one starting label corresponding to the label of the root and of productions encoding the way labels spread in the generating tree. As we explain in [6], the sequence enumerating the class $\mathcal{C}$ can be recovered from the succession rule itself, without reference to the specifics of the objects in $\mathcal{C}$: indeed, the nth term of the sequence is the total number of labels (counted with repetition) that are produced from the root by $n-1$ applications of the productions, or equivalently, the number of nodes at distance n from the root in the generating tree.

From the growth of Dyck paths described above, we classically obtain the following succession rule associated with Catalan numbers (corresponding to the tree shown in Figure 1, right):

$$\Omega_{Cat} = \begin{cases} (1) \\ (k) \rightsquigarrow (1), (2), \dots, (k), (k+1). \end{cases}$$

The intended meaning of the label (k) is the number of D steps in the last descent of a path.

The class of Dyck paths is just one of many families of discrete objects counted by the Catalan numbers (sequence A000108 in [12]).

2 A hierarchy of inversion sequences

In this paper we consider a hierarchy of families of inversion sequences ordered by inclusion according to the following scheme:

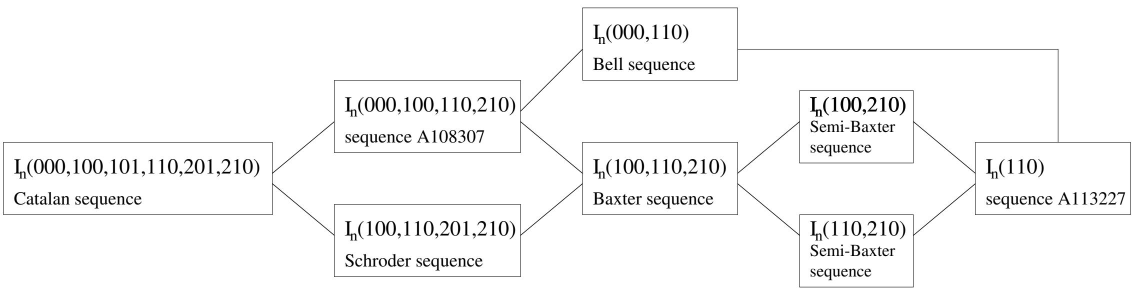

Figure 2: The families of inversion sequences considered in the paper, ordered by inclusion.

The aim of this paper is to handle all these families in a unified way by providing, for each of them: a (possible) combinatorial characterization; a recursive growth by means of generating trees; enumeration; possible connections with other combinatorial structures. In most of these cases we prove enumerative results conjectured in [14].

The recursive construction (and the generating tree) of any family in the scheme of Fig. 2 is obtained as an extension of the construction (and the generating tree) of a smaller one, starting from the family $I_n(\geq, -, \geq)$ defining Catalan numbers. We also point out that our paper provides a natural instance of the inclusion "*Catalan in Schröder in Baxter in semi-Baxter*". As pointed out in [6, 7] these inclusions are obvious on pattern-avoiding permutations, but they remain quite obscure on other objects.

2.1 $I_n(000, 100, 101, 110, 201, 210)$: Catalan sequence.

We start considering the family $I_n^C = I_n(\geq, -, \geq) = I_n(000, 100, 101, 110, 201, 210)$. These inversion sequences can be characterized as follows: a sequence $(e_1 \dots e_n) \in I_n^C$ if and only if for any i we have: if $e_{i+1} < e_i$ then $e_j > e_i$ for all $j > i+1$. Our main results concern:

1. $|I_n^C| = c_n$, i.e. the nth Catalan number, as conjectured in [14], by providing a recursive growth of the family according to Ω_{Cat}.

2. a bijectiion between I_n^C and non crossing partitions of size n, which are a well known Catalan structure.

3. a recursive growth of I_n^C according to the generating tree:

$$\Omega_{Cat'} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (0,k+1)^h \\ \qquad\qquad (h+k,1), \dots, (h+1,k). \end{cases}$$

   This is a new generating tree for Catalan numbers. Observe that we need not prove it since it is implied by the above steps. All the families in our scheme (unless otherwise specified) have a grow which extends the one provided by $\Omega_{Cat'}$.

4. the proof that I_n^{Cat} is just the set of inversion sequences of AV_n(12-3, 2-14-3), which therefore turns out to be another family of permutations counted by Catalan numbers.

2.2 $I_n(100, 110, 201, 210)$: Schröder sequence

Let $I_n^S = I_n(\geq, -, >) = I_n(100, 110, 201, 210)$. These inversion sequences can be characterized as follows: for every i, j with $i < j$ such that $e_i \geq e_j$, for all $k > j$ we have $e_k \geq e_i$. In [14] is proved that $I_n^S = s_n$, i.e. the nth Schröder number (sequence A006318 in [12]) by a standard decomposition technique. We prove that I_n^S grow according to

$$\Omega_{Sc} = \begin{cases} (1,1) \\ (h,k) \quad \rightsquigarrow \quad (1,k+1)^h \\ \qquad\qquad\qquad (h+1,k), \dots, (h+k,1)\,, \end{cases}$$

which is a clear extension of $\Omega_{Cat'}$. Observe that this is a new generating tree for Schröder numbers, different from the ones given in [2, 6, 16].

2.3 $I_n(000, 100, 110, 210)$: sequence A108307

Let $I_n^A = I_n(\geq, \geq, \geq) = I_n(000, 100, 110, 210)$. These inversion sequences can be characterized as follows: for any three indices $i < j < k$, if $e_i \leq e_j$, then $e_k > e_j$. This implies that every such inversion sequence can be uniquely decomposed in two strictly increasing sequences. For instance, $0113236567 \in I_{10}^A$ and taking its left-to-right maxima it can be decomposed in 01367 and 12356, which are strictly increasing sequences. We prove that I_n^A grows according to

$$\Omega_A = \begin{cases} (1,1) \\ (h,k) \quad \rightsquigarrow \quad (0,k+1), \dots, (h-1,k+1) \\ \qquad\qquad\qquad (h+1,k), \dots, (h+k,1)\,, \end{cases}$$

which is also a clear extension of $\Omega_{Cat'}$. Using standard techniques, from Ω_A we obtain a functional equation satisfied by the generating function of I^A. Applying some variants of the kernel method and then the Lagrange inversion formula we have proved that the numbers $b_n = I_n^A$ have a D-finite generating function and satisfy the following polynomial recurrence relation:

$$8(n+3)(n+2)(n+1)\, b_n \;+\; (n+2)(15n^2+133n+280)\, b_{n+1} \;+\; (92n^2+6n^3+464n+776)\, b_{n+2} \\ -\,(n+9)(n+8)(n+6)\, b_{n+3} \;=\; 0\,. \tag{1}$$

In [14] the authors conjecture that $\{b_n\}_{n\geq 0}$ is sequence A108307 in [12]. This sequence counts, among the others:

1. partitions avoiding enhanced 3-crossings. These objects have been studied in [5], where the authors proved that their generating function is D-finite and determined a recursive formula for the numbers a_n of partitions avoiding enhanced 3-crossings of size n:

$$\begin{aligned}&8(n+3)(n+2)(n+1)a_n + 3(n+2)(5n^2+47n+104)a_{n+1} + 3(n+4)(2n+11)(n+7)a_{n+2}\\&-(n+9)(n+8)(n+7)a_{n+3}=0.\end{aligned} \tag{2}$$

2. inversion sequences defined as follows: $e_0 = 0, 0 \leq e_1 \leq 1$, $e_n \leq \max\{e_{n-1}, e_{n-2}\} + 1$.

Using standard techniques on the two recurrence relations (1) and (2) we have been able to prove that $a_n = b_n$ for all $n \geq 1$, thus proving the conjecture in [14].

2.4 $I_n(100, 110, 210)$: Baxter sequence

Let $I_n^B = I_n(\geq, \geq, >) = I_n(100, 110, 210)$. These inversion sequences can be characterized as follows: for any three indices i, j with $i < j$ such that $e_i \geq e_j$, for any $k > j$ we have $e_k > e_j$ or $e_k \geq e_i$. We prove that I_n^B grows according to

$$\Omega_{Bax} = \begin{cases} (1,1) & \\ (h,k) & \rightsquigarrow \quad (1,k+1),\ldots,(h-1,k+1),(1,k+1) \\ & \qquad (h+1,k),\ldots,(h+k,1)\,. \end{cases}$$

In [14] it is conjectured that I_n^B is counted by Baxter numbers (sequence A001181 in [12]). The generating tree Ω_{Bax} is not known in the literature, so in order to prove this conjecture we have solved the functional equation arising from Ω_{Bax}, applying the kernel method and then the Lagrange inversion formula. Then we have obtained a formula for the cardinality of I_n^B:

$$\begin{aligned}&2\sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k+1}\binom{n-1}{k-2}+\sum_{p=0}^{n}\left[\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-p}\binom{n}{n+p-k+1}+\frac{p}{n}\binom{n}{n+p-k-1}\right]\\&-2\left[\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-p}\binom{n}{n+p-k+2}+\frac{p}{n}\binom{n}{n+p-k-2}\right]+\left[\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k-p}\binom{n}{n+p-k+3}+\frac{p}{n}\binom{n}{n+p-k-3}\right].\end{aligned}$$

but we have not been able to prove that this formula gives Baxter numbers, although we have checked that the two sequences coincide for a huge amount of terms. On the other side, we have not been able to find a growth of a Baxter object (as, for instance, Baxter permutations or Baxter slicings [6]) according to Ω_{Bax}.

2.5 $I_n(110, 210)$: Semi-Baxter sequence

Here we consider the inversion sequences $I_n(\geq, >, >) = I_n(110, 210)$ and $I_n(>, \geq, >) = I_n(100, 210)$, which in [14] are both conjectured to be counted by the sequence of *semi-Baxter numbers* (sequence A117106 in [12]).

We prove this conjecture by using a family of permutations counted by sequence A117106, called *semi-Baxter permutations* [7] and defined by the avoidance of the pattern 2-41-3 (recall that Baxter permutations are defined by the avoidance of the same pattern, and 3-14-2). In [7] the problem of enumerating semi-Baxter permutations is solved, pushing further the techniques that were used to enumerate Baxter permutations in [3]. A generating tree with two labels for semi-Baxter permutations was provided,

$$\Omega_{Semi} = \begin{cases} (1,1) & \\ (h,k) \rightsquigarrow & (1,k+1),\ldots,(h,k+1) \\ & (h+k,1),\ldots,(h+1,k). \end{cases}$$

Then, the functional equation associated with it was solved using variants of the kernel method [3], leading to the proof that the n-th semi-Baxter number sb_n satisfies a simple recurrence formula, where $sb_0 = 1$, $sb_1 = 1$, and for all $n \geq 2$,

$$sb_n = \frac{11n^2 + 11n - 6}{(n+4)(n+3)} sb_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)} sb_{n-2}.$$

Recall that semi-Baxter numbers count also 2-14-3-avoiding permutations (*plane permutations*), introduced in [4], whose growth is governed by the rule Ω_{Semi}.

Here, we prove that both $I_n(110, 210)$ and $I_n(100, 210)$ grow according to Ω_{Semi} thus proving that they are counted by semi-Baxter numbers, as conjectured in [14]. Observe that Ω_{Semi} extends Ω_{Bax}.

2.6 $I_n(000, 110)$: Bell sequence

It is worth spending a few words on $I_n(000, 110)$, which is conjectured to be enumerated by Bell numbers [14], as well as set partitions of size n. We provide a proof of this fact by showing that $I_n(000, 110)$ grows according to the known generating tree

$$\Omega_B = \left\{ \begin{array}{l} (2) \\ (h) \rightsquigarrow (h)^{h-1}(h+1) \end{array} \right.$$

defining Bell numbers (see, for instance [1]). Relying on this growth, we build up a direct bijection between inversion sequences of $I_n(000, 110)$ and set partitions of size n, which extends the bijection (in Section 2.1) between I_n^C and non-crossing partitions of size n. Unfortunately the construction given by Ω_B is not an extension of Ω_{semi}.

2.7 $I_n(110)$: sequence A113227

We consider $I_n^L = I_n(110) = I_n(=, >, >)$. These sequences can be characterized as those inversion sequences such that, with $i > 1$, e_i satisfies: $e_i > Max(e_1 \dots e_{i-1})$ or $e_i \geq Max_2(e_1 \dots e_{i-1})$, where $Max_2(e_1 \dots e_i) = Max\{e_{j_1} | \exists j_2 \neq j_1, e_{j_1} = e_{j_2}\}$.

In [9] the authors prove that the number $p_n = |I_n^L|$ can be expressed as $p_n = \sum_j p_{n,j}$, where the terms $p_{n,j}$ satisfy the recurrence relation

$$\left\{ \begin{array}{l} p_{1,1} = 1, \\ p_{n,j} = p_{n-1,j-1} + j \sum_{i=j}^{n-1} p_{n-1,i} \,. \end{array} \right. \tag{3}$$

Thus, $\{p_n\}_{n \geq 0}$ is sequence A113227 in [12]. This sequence has been studied by D. Callan in [8], and it is proved to count several families of objects, such as *marked valleys Dyck paths*, *increasing ordered trees with increasing leaves* or *permutations avoiding the generalized pattern* 1-23-4.

In this paper we prove that I_n^L grows according to the generating tree:

$$\Omega_L = \left\{ \begin{array}{l} (2) \\ (h) \rightsquigarrow (1)(2)^2 \dots (h)^h (h+1) \end{array} \right.$$

which is a clear extension of Ω_{Cat}, but is not related to the other considered generating trees. We also prove that sequence A113227 counts a family of lattice paths (called *steady paths*) which extend Dyck paths and are easily bijective to permutations avoiding 1-34-2. Moreover, we determine a recursive growth of these paths according to the rule

$$\Omega_{L'} = \left\{ \begin{array}{lll} (0,2) & & \\ (h,k) & \rightsquigarrow & (0,k+1), \dots, (0,h+k+1) \\ & & (h+k-1,2), \dots, (h+1,k) \,. \end{array} \right.$$

Unfortunately we have not been able to find a bijection between these steady paths (or permutations avoiding 1-34-2) and I_n^L, nor to determine a recursive growth of I_n^L according to $\Omega_{L'}$.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps, *Generating functions for generating trees*, Disc. Math., 246:29–55, 2002.

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, *ECO: a methodology for the Enumeration of Combinatorial Objects*, J. Diff. Eq. and App., 5:435–490, 1999.

[3] M. Bousquet-Mélou, *Four classes of pattern-avoiding permutations under one roof: generating trees with two labels*, Electron. J. Combin., 9(2), article R19, 2003.

[4] M. Bousquet-Mélou, S. Butler, *Forest-like permutations*, Ann. Combin., 11:335–354, 2007.

[5] M. Bousquet-Mélou, G. Xin, On partitions avoiding 3-crossings, arXiv:math/0506551 [math.CO], 2005-2006.

[6] M. Bouvel, V. Guerrini, S. Rinaldi, *Slicings of parallelogram polyominoes, or how Baxter and Schröder can be reconciled*, Proceedings of FPSAC 2016, DMTCS proc. BC, 287–298, 2016.

[7] M. Bouvel, V. Guerrini, A. Rechnitezer, S. Rinaldi, Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence, `arXiv:1702.04529 [math.CO]`.

[8] D. Callan, A bijection to count (1-23-4)-avoiding permutations, online available on *Arxiv1008.2375.v1*.

[9] S. Corteel, M. A. Martinez, C. D. Savage, M. Weselcouch, Patterns in inversion sequences I, *Discrete Mathematics and Theoretical Computer Science*, 18(2), 2016.

[10] F.R.K. Chung, R. Graham, V. Hoggatt, M. Kleiman, *The number of Baxter permutations*, J. Comb. Theory A, 24(3):382–394, 1978.

[11] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[12] OEIS Foundation Inc., *The On-line Encyclopedia of Integer Sequences*, `http://oeis.org`, 2011.

[13] T. Mansour, M. Shattuck, Pattern avoidance in inversion sequences, *Pure Mathematics and Applications*, 25(2):157–176, 2015.

[14] M. A. Martinez, C. D. Savage, *Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations*, Arxiv preprint, `1609.08106`, 2016.

[15] L. Pudwell, *Enumeration schemes for permutations avoiding barred patterns*, Electron. J. Combin., 17(1), article R29, 2010.

[16] J. West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math., 146:247–262, 1995.