1 Introduction

An inversion sequence is an integer sequence \((e_1, \ldots, e_n)\) satisfying \(0 \leq e_i < i\) for all \(i = 1, \ldots, n\). There is a natural bijection \(\Theta : S_n \rightarrow I_n\) from \(S_n\), the set of permutations of length \(n\), to \(I_n\), the set of inversion sequences of length \(n\). Under this bijection, \(e = \Theta(\pi)\) is obtained from a permutation \(\pi = \pi_1 \ldots \pi_n \in S_n\) by setting \(e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|\). The study of patterns in inversion sequences was introduced in [9] and [13]. In [9] the authors study inversion sequences that avoid words of length 3, while in [13] inversion sequences avoiding permutations of length 3 are considered. For instance, the inversion sequences that avoid the pattern 110 are those that avoid words of length 3, while in [13] inversion sequences avoiding permutations of length 3 are considered. For instance, the inversion sequences that avoid the pattern 110 are those with no \(i < j < k\) such that \(e_i = e_j > e_k\). This study has been carried on in [14], where the notion of pattern avoidance is generalized to a triple of binary relations \((\rho_1, \rho_2, \rho_3)\) and study the set \(I_n(\rho_1, \rho_2, \rho_3)\) of those \(e \in I_n\) with no \(i < j < k\) such that \(e_i \rho_1 e_j, e_j \rho_2 e_k, \text{ and } e_i \rho_3 e_k\). For example \(I_n(=, >, >) = I_n(110)\). In [14] all triples of relations of the set \(\{<, >, \leq, \geq, =, \neq\}\) are studied, where \(\neq\) on a set \(S\) is the cartesian product, i.e. \(-S \times S\). Therefore all 343 patterns are considered and partitioned in 98 equivalence classes. In [14] the authors found several enumeration results beyond those in [9, 13], and formulated several conjectures, some of which are studied in this paper.

Generating trees and succession rules will also be important for our work. We give a brief general presentation below. Details can be found for instance in [1, 2, 3, 16]. We also review the classical generating tree for Dyck paths and the classical succession rule for Catalan numbers.

Consider any combinatorial class \(\mathcal{C}\), that is to say any set of discrete objects equipped with a notion of size, such that there is a finite number of objects of size \(n\) for any integer \(n\). Assume also that \(\mathcal{C}\) contains exactly one object of size 1. A generating tree for \(\mathcal{C}\) is an infinite rooted tree, whose vertices are the objects of \(\mathcal{C}\), each appearing exactly once in the tree, and such that objects of size \(n\) are at distance \(n\) from the root (with the convention that the root is at distance 1 from itself). The children of some object \(c \in \mathcal{C}\) are obtained by adding an atom (i.e. a piece of object that makes its size increase by 1) to \(c\). Of course, since every object should appear only once in the tree, not all additions are possible. We should ensure the unique appearance property by considering only additions that follow some restricted rules. We will call the growth of \(\mathcal{C}\) the process of adding atoms following these prescribed rules.

Such a growth for Dyck paths is described in [2]. Recall that a Dyck path of semilength \(n\) is a lattice path using up \(U\) and down \(D\) unit steps and running from \((0,0)\) to \((2n,0)\) remaining weakly above the \(x\)-axis. The atoms are peaks, that is to say \(UD\) factors. To ensure that all Dyck paths appear exactly once in the generating tree, peaks are inserted in the last descent of the path, which is the longest suffix containing only the letter \(D\). More precisely, the children of a Dyck path \(w \cdot UD^k\) are \(w \cdot UDD^k, w \cdot UdUDD^{k-1}, \ldots, w \cdot UD^{k-1}DD\) and \(w \cdot UD^k UD\). The first few levels of the generating tree for Dyck paths are shown in Figure 1 (left).

Since the growth of \(\mathcal{C}\) uniquely defines the shape of the generating tree, for enumerative purpose we choose to identify a generating tree with its shape without regarding the objects that label its nodes. If such growth is particularly regular, we can represent it via a succession rule. A succession

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rule is made of one starting label corresponding to the label of the root and of productions encoding
the way labels spread in the generating tree. As we explain in [6], the sequence enumerating the
class $\mathcal{C}$ can be recovered from the succession rule itself, without reference to the specifics of the
objects in $\mathcal{C}$: indeed, the $n$th term of the sequence is the total number of labels (counted with
repetition) that are produced from the root by $n-1$ applications of the productions, or equivalently,
the number of nodes at distance $n$ from the root in the generating tree.

From the growth of Dyck paths described above, we classically obtain the following succession
rule associated with Catalan numbers (corresponding to the tree shown in Figure 1, right):

$$\Omega_{\text{Cat}} = \{(1) \rightarrow (1), (2), \ldots, (k), (k+1)\}.$$ 

The intended meaning of the label $(k)$ is the number of $D$ steps in the last descent of a path.

The class of Dyck paths is just one of many families of discrete objects counted by the Catalan
numbers (sequence A000108 in [12]).

2 A hierarchy of inversion sequences

In this paper we consider a hierarchy of families of inversion sequences ordered by inclusion ac-
cording to the following scheme:

![Diagram of inversion sequences hierarchy]

The aim of this paper is to handle all these families in a unified way by providing, for each
of them: a (possible) combinatorial characterization; a recursive growth by means of generating
trees; enumeration; possible connections with other combinatorial structures. In most of these
cases we prove enumerative results conjectured in [14].

The recursive construction (and the generating tree) of any family in the scheme of Fig. 2 is
obtained as an extension of the construction (and the generating tree) of a smaller one, starting
from the family $L_n(\ge, -, \ge)$ defining Catalan numbers. We also point out that our paper provides
a natural instance of the inclusion “Catalan in Schröder in Baxter in semi-Baxter”. As pointed
out in [6, 7] these inclusions are obvious on pattern-avoiding permutations, but they remain quite
obscure on other objects.
2.1 $I_n(000, 100, 101, 110, 201, 210)$: Catalan sequence.

We start considering the family $I_n^C = I_n(\geq, -, \geq) = I_n(000, 100, 101, 110, 201, 210)$. These inversion sequences can be characterized as follows: a sequence $(e_1 \ldots e_n) \in I_n^C$ if and only if for any $i$ we have: if $e_{i+1} < e_i$ then $e_j > e_i$ for all $j > i + 1$. Our main results concern:

1. $|I_n^C| = C_n$, i.e. the $n$th Catalan number, as conjectured in [14], by providing a recursive growth of the family according to $\Omega_{Cat}$.

2. a bijection between $I_n^C$ and non crossing partitions of size $n$, which are a well known Catalan structure.

3. a recursive growth of $I_n^C$ according to the generating tree:

$$\Omega_{Cat'} = \left\{ \begin{array}{l}
(1,1) \\
(h,k) \rightsquigarrow (0,k+1)^h \\
(h+k,1), \ldots, (h+1,k),
\end{array} \right. $$

This is a new generating tree for Catalan numbers. Observe that we need not prove it since it is implied by the above steps. All the families in our scheme (unless otherwise specified) have a growth which extends the one provided by $\Omega_{Cat'}$.

4. the proof that $I_n^{Cat}$ is just the set of inversion sequences of $AV'_n(12-3, 2-14-3)$, which therefore turns out to be another family of permutations counted by Catalan numbers.

2.2 $I_n(100, 110, 201, 210)$: Schröder sequence

Let $I_n^S = I_n(\geq, -, \geq) = I_n(100, 110, 201, 210)$. These inversion sequences can be characterized as follows: for every $i,j$ with $i < j$ such that $e_i \geq e_j$, for all $k > j$ we have $e_k \geq e_i$. In [14] is proved that $I_n^S = s_n$, i.e. the $n$th Schröder number (sequence A006318 in [12]) by a standard decomposition technique. We prove that $I_n^S$ grow according to

$$\Omega_{Sc} = \left\{ \begin{array}{l}
(1,1) \\
(h,k) \rightsquigarrow (1,k+1)^h \\
(h+1,k), \ldots, (h+k,1),
\end{array} \right. $$

which is a clear extension of $\Omega_{Cat'}$. Observe that this is a new generating tree for Schröder numbers, different from the ones given in [2, 6, 16].

2.3 $I_n(000, 100, 110, 210)$: sequence A108307

Let $I_n^A = I_n(\geq, \geq, \geq) = I_n(000, 100, 110, 210)$. These inversion sequences can be characterized as follows: for any three indices $i < j < k$, if $e_i \leq e_j$, then $e_k > e_j$. This implies that every such inversion sequence can be uniquely decomposed in two strictly increasing sequences. For instance, 0113236567 $\in I^n_1$ and taking its left-to-right maxima it can be decomposed in 01367 and 12356, which are strictly increasing sequences. We prove that $I_n^A$ grows according to

$$\Omega_A = \left\{ \begin{array}{l}
(1,1) \\
(h,k) \rightsquigarrow (0,k+1), \ldots, (h-1,k+1) \\
(h+1,k), \ldots, (h+k,1),
\end{array} \right. $$

which is also a clear extension of $\Omega_{Cat'}$. Using standard techniques, from $\Omega_A$ we obtain a functional equation satisfied by the generating function of $I^n_A$. Applying some variants of the kernel method and then the Lagrange inversion formula we have proved that the numbers $b_n = I^n_A$, have a D-finite generating function and satisfy the following polynomial recurrence relation:

$$8(n+3)(n+2)(n+1)b_n + (n+2)(15n^2 + 133n + 280)b_{n+1} + (92n^2 + 6n^3 + 464n + 776)b_{n+2} - (n+9)(n+8)(n+6)b_{n+3} = 0. $$

(1)
In [14] the authors conjecture that \( \{b_n\}_{n \geq 0} \) is sequence A108307 in [12]. This sequence counts, among the others:

1. partitions avoiding enhanced 3-crossings. These objects have been studied in [5], where the authors proved that their generating function is D-finite and determined a recursive formula for the numbers \( a_n \) of partitions avoiding enhanced 3-crossings of size \( n \):

\[
8(n+3)(n+2)(n+1)a_n + 3(n+2)(5n^2 + 47n + 104)a_{n+1} + 3(n+4)(2n+1)(n+7)a_{n+2} \\
- (n+9)(n+8)(n+7)a_{n+3} = 0.
\]

(2)

2. inversion sequences defined as follows: \( e_0 = 0, 0 \leq e_1 \leq 1, e_n = \max\{e_{n-1}, e_{n-2}\} + 1 \).

Using standard techniques on the two recurrence relations (1) and (2) we have been able to prove that \( a_n = b_n \) for all \( n \geq 1 \), thus proving the conjecture in [14].

### 2.4 \( I_n(100, 110, 210) \): Baxter sequence

Let \( I^B_n = I_n(\geq, \geq, >) = I_n(100, 110, 210) \). These inversion sequences can be characterized as follows: for any three indices \( i, j \) with \( i < j \) such that \( e_i \geq e_j \), for any \( k > j \) we have \( e_k > e_j \) or \( e_k \geq e_i \). We prove that \( I^B_n \) grows according to

\[
\Omega_{Bax} = \left\{ \begin{array}{ll} 
(1,1) \\
(h,k) \Rightarrow (h+1,k), \ldots, (h+k,1), (1,k+1) \\
(1,k+1) \ldots, (h+1,k), (h+k,1) \end{array} \right. 
\]

In [14] it is conjectured that \( I^B_n \) is counted by Baxter numbers (sequence A001181 in [12]). The generating tree \( \Omega_{Bax} \) is not known in the literature, so in order to prove this conjecture we have solved the functional equation arising from \( \Omega_{Bax} \), applying the kernel method and then the Lagrange inversion formula. Then we have obtained a formula for the cardinality of \( I^B_n \):

\[
2 \sum_{k=0}^n \frac{1}{n} \binom{n}{k+1} \binom{n-1}{k-2} + \sum_{p=0}^n \left[ \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+p-k+1}{k-p}} \left( \frac{\binom{n}{k-p}}{\binom{n+p-k-1}{k-p}} \right) + \frac{p}{n} \frac{\binom{n}{k-p}}{\binom{n+p-k-2}{k-p}} \right] \\
- 2 \left[ \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+p-k+2}{k-p}} + \frac{p}{n} \frac{\binom{n}{k-p}}{\binom{n+p-k-2}{k-p}} + \frac{p}{n} \frac{\binom{n}{k-p}}{\binom{n+p-k-3}{k-p}} \right].
\]

but we have not been able to prove that this formula gives Baxter numbers, although we have checked that the two sequences coincide for a huge amount of terms. On the other side, we have not been able to find a growth of a Baxter object (as, for instance, Baxter permutations or Baxter slicings [6]) according to \( \Omega_{Bax} \).

### 2.5 \( I_n(110, 210) \): Semi-Baxter sequence

Here we consider the inversion sequences \( I_n(\geq, >) = I_n(110, 210) \) and \( I_n(>, \geq, >) = I_n(100, 210) \), which in [14] are both conjectured to be counted by the sequence of semi-Baxter numbers (sequence A117106 in [12]).

We prove this conjecture by using a family of permutations counted by sequence A117106, called semi-Baxter permutations [7] and defined by the avoidance of the pattern 2-41-3 (recall that Baxter permutations are defined by the avoidance of the same pattern, and 3-14-2). In [7] the problem of enumerating semi-Baxter permutations is solved, pushing further the techniques that were used to enumerate Baxter permutations in [3]. A generating tree with two labels for semi-Baxter permutations was provided,

\[
\Omega_{Semi} = \left\{ \begin{array}{ll} 
(1,1) \\
(h,k) \Rightarrow (h+1,k), \ldots, (h+k,1) \\
(1,k+1) \ldots, (h+1,k) \end{array} \right. 
\]
Then, the functional equation associated with it was solved using variants of the kernel method [3], leading to the proof that the \( n \)-th semi-Baxter number \( sb_n \) satisfies a simple recurrence formula, where \( sb_0 = 1 \), \( sb_1 = 1 \), and for all \( n \geq 2 \),
\[
  sb_n = \frac{11n^2 + 11n - 6}{(n + 4)(n + 3)} sb_{n-1} + \frac{(n - 3)(n - 2)}{(n + 4)(n + 3)} sb_{n-2}.
\]

Recall that semi-Baxter numbers count also 2-14-3-avoiding permutations (\emph{plane permutations}), introduced in [4], whose growth is governed by the rule \( \Omega_{\text{Semi}} \).

Here, we prove that both \( I_n(110, 210) \) and \( I_n(100, 210) \) grow according to \( \Omega_{\text{Semi}} \) thus proving that they are counted by semi-Baxter numbers, as conjectured in [14]. Observe that \( \Omega_{\text{Semi}} \) extends \( \Omega_{\text{Bax}} \).

\subsection{2.6 \( I_n(000, 110) \): Bell sequence}

It is worth spending a few words on \( I_n(000, 110) \), which is conjectured to be enumerated by Bell numbers [14], as well as set partitions of size \( n \). We provide a proof of this fact by showing that \( I_n(000, 110) \) grows according to the known generating tree
\[
  \Omega_B = \left\{ \begin{array}{l}
  (2) \\
  (h) \leadsto (h)^{h-1}(h + 1)
\end{array} \right. 
\]

defining Bell numbers (see, for instance [1]). Relying on this growth, we build up a direct bijection between inversion sequences of \( I_n(000, 110) \) and set partitions of size \( n \), which extends the bijection (in Section 2.1) between \( I_n^{\ell} \) and non-crossing partitions of size \( n \). Unfortunately the construction given by \( \Omega_B \) is not an extension of \( \Omega_{\text{semi}} \).

\subsection{2.7 \( I_n(110) \): sequence A113227}

We consider \( I_n^L = I_n(110) = I_n(=, >, >) \). These sequences can be characterized as those inversion sequences such that, with \( i > 1 \), \( e_i \) satisfies: \( e_i > \max(e_1 \ldots e_{i-1}) \) or \( e_i \geq \max\{e_1 \ldots e_{i-1}\} \), where \( \max\{e_1 \ldots e_i\} = \max\{e_j, \exists j_2 \neq j_1, e_{j_1} = e_{j_2}\} \).

In [9] the authors prove that the number \( p_n = |I_n^L| \) can be expressed as \( p_n = \sum_j p_{n,j} \), where the terms \( p_{n,j} \) satisfy the recurrence relation
\[
\begin{array}{l}
  p_{1,1} = 1, \\
  p_{n,j} = p_{n-1,j-1} + j \sum_{i=1}^{n-1} p_{n-1,i}.
\end{array}
\]

Thus, \( \{p_n\}_{n \geq 0} \) is sequence A113227 in [12]. This sequence has been studied by D. Callan in [8], and it is proved to count several families of objects, such as \emph{marked valleys Dyck paths, increasing ordered trees with increasing leaves or permutations avoiding the generalized pattern 1-23-4}.

In this paper we prove that \( I_n^L \) grows according to the generating tree:
\[
  \Omega_L = \left\{ \begin{array}{l}
  (2) \\
  (h) \leadsto (1)(2)^2 \ldots (h)^h(h + 1)
\end{array} \right. 
\]

which is a clear extension of \( \Omega_{\text{Cat}} \), but is not related to the other considered generating trees. We also prove that sequence A113227 counts a family of lattice paths (called \emph{steady paths}) which extend Dyck paths and are easily bijective to permutations avoiding 1-34-2. Moreover, we determine a recursive growth of these paths according to the rule
\[
  \Omega_{L'} = \left\{ \begin{array}{l}
  (0, 2) \\
  (h, k) \leadsto (0, k + 1), \ldots, (0, h + k + 1) \\
  (h + k - 1, 2), \ldots, (h + 1, k).
\end{array} \right.
\]

Unfortunately we have not been able to find a bijection between these steady paths (or permutations avoiding 1-34-2) and \( I_n^L \), nor to determine a recursive growth of \( I_n^L \) according to \( \Omega_{L'} \).
References


[8] D. Callan, A bijection to count (1-23-4)-avoiding permutations, online available on Arxiv1008.2375.v1.


