Finding functional equations for two 2 by 4 classes

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In recent years, much progress has been made on enumerating two-by-four permutation classes [2, 3, 4]. Of the thirtyeight two-by-four Wilf-equivalence classes, all but five have been enumerated. Of these five, three have been analyzed extensively by Albert et al. [1], and have been conjectured not to have differentially finite generating functions. In this talk, we discuss the remaining two classes, $\mathrm{Av}(\mathbf{3412}, \mathbf{2413})$ and $\mathrm{Av}(\mathbf{1432}, \mathbf{2143})$.

For $\mathrm{Av}(\mathbf{3412}, \mathbf{2413})$, we find two functional equations relating the generating function for the whole class to that of skew-indecomposable permutations within the class. Let $\mathcal{F}(z,t)$ represent the generating function for permutations in the whole class, and $\mathcal{F}_s(z,t)$ represent skew-indecomposable permutations within the class, where z marks the length of the permutation and t is the position of the first increase (or the length of the initial decreasing segment). Then

$$\mathcal{F}(z,t) = \frac{\mathcal{F}_s(z,t)}{(1-zt)(1-z)} + \frac{1}{1-zt}\,, \tag{1}$$

since a permutation which avoids $\mathbf{3412}$ either have zero or one nontrivial skew-component.

The second equation is given by

$$\mathcal{F}_s(z,t) = z\mathcal{F}(z,1)(\mathcal{F}(z,t)-1) + \frac{z(\mathcal{F}(z,t)-1)}{(1-z)(\mathcal{F}(z,1)-t)}\left[\mathcal{F}(z,1)\mathcal{F}_s(z,\mathcal{F}(z,1)) - t\mathcal{F}_s(z,t)\right]. \tag{2}$$

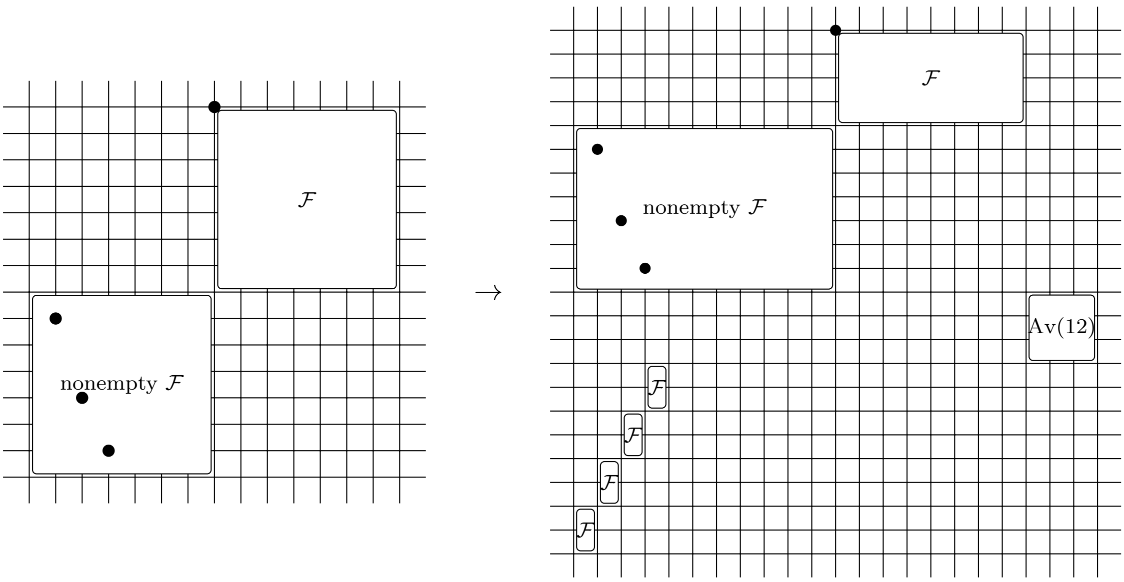

Figure 1: Adding a lower layer to a skew-indecomposable permutation in $\mathcal{F}$, where $\mathcal{F}$ represents $\mathrm{Av}(\mathbf{3412}, \mathbf{2413})$.

In Figure 1 we show the logic behind this second equation. Permutations in $\mathcal{F}_s$ are generated by a unique sequence of operations consisting of adding a *lower layer*, where a lower layer consists of a nonempty decreasing sequence of rightmost minima, followed by at least one nonempty permutation in $\mathcal{F}$ as a new minimum to the left of the permutation's maximum.

For Av($\mathbf{1432}, \mathbf{2143}$), we use similar logic to find a functional equation by utilizing the structure of the permutations. We let $\mathcal{G}(z,t)$ represent the generating function for permutations in the whole class, where z measures length and t measures the number of nontrivial allowable positions for new left-to-right minima. Our functional equation for $\mathcal{G}(z,t)$ has terms of the form $\mathcal{G}(z, \frac{1}{1-z})$, as well as $\mathcal{G}_t(z,1)$.

For each of these two classes, the functional equations allow us to generate initial terms, though explicit formulas for the generating functions themselves remain elusive. Specifically, the terms $\mathcal{F}_s(z, \mathcal{F}(z,1))$ and $\mathcal{G}(z, \frac{1}{1-z})$ are the sticking points. We plan to analyze these equations more in the future.

This talk is based on joint work with Jay Pantone.

References

[1] M. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter, Generating permutations with restricted containers, arXiV preprint arXiv:1510.00269, 2015.

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[4] S. Miner, Enumeration of several two-by-four classes, arXiV preprint arXiv:1610.01908, 2016.