THE PINNACLE SET OF A PERMUTATION

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1. Introduction

Let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$, which we will always write as words, $w = w(1)w(2)\cdots w(n)$. A *peak* is a descent that is preceded by an ascent, and the *peak set* of a permutation w, denoted $\mathrm{Pk}(w)$, is the collection of its peaks,

$$\mathrm{Pk}(w) = \{i : w(i-1) < w(i) > w(i+1)\}.$$

Any subset of $\{1, 2, \ldots, n-1\}$ is the descent set of some permutation in S_n, but the same cannot be said for peak sets. For example, peaks cannot occur in the first or last positions of a permutation, so $\mathrm{Pk}(w) \subseteq \{2, \ldots, n-1\}$ for any $w \in S_n$. Recently Billey, Burdzy, and Sagan [1] asked about how many permutations in S_n have a given peak set. One of their results is that for a fixed set S, the number of $w \in S_n$ for which $\mathrm{Pk}(w) = S$ is a power of two times a polynomial in n, and they give techniques for explicit computation of this polynomial in special cases. In this abstract, we will study questions related to peaks, but rather than tracking peaks by their indices, we use their values.

Definition 1.1. A *pinnacle* of a permutation w is a value $w(i)$ such that $w(i-1) < w(i) > w(i+1)$. The *pinnacle set* of w is

$$\mathrm{Pin}(w) = \{w(i) : i \in \mathrm{Pk}(w)\}.$$

Certainly $|\,\mathrm{Pk}(w)| = |\,\mathrm{Pin}(w)|$, but the sets themselves need not be the same: if $w = 315264$, then $\mathrm{Pk}(w) = \{3, 5\}$ and $\mathrm{Pin}(w) = \{5, 6\}$. The introduction of pinnacle sets lead to questions about the value

$$p_S(n) := |\{w \in S_n : \mathrm{Pin}(w) = S\}|.$$

We remark that descent topsets, which are defined similarly to but are distinct from pinnacle sets, and related ideas have appeared sporadically in the literature on permutation statistics, e.g., see [2, 3, 4, 5, 6, 7]. The question of enumeration by pinnacle sets does not appear to have been addressed in the literature.

The questions we address are the following.

1. When is $p_S(n) > 0$? That is, when is a set S the pinnacle set of some permutation in S_n?
2. Given a pinnacle set $S \subseteq [n]$, how do we compute $p_S(n)$?
3. For a given n, what choice of $S \subseteq [n]$ maximizes or minimizes $p_S(n)$?

In Section 2 we identify conditions under which a set S is the pinnacle set for some permutation, fully answering Question 1. In Section 3 we develop both a quadratic and a linear recurrence for $p_S(n)$, which partially answers Question 2. Further, we identify bounds for $p_S(n)$, answering Question 3.

* Partially supported by Simons Foundation Collaboration Grants for Mathematicians.

2. Admissible pinnacle sets

Not every set is the peak set of a permutation. Likewise, not every set is a pinnacle set. For one thing, each peak must have a non-peak on each side of it, so the number of peaks must be strictly less than half the number of letters in the permutation, hence $n > 2|\operatorname{Pk}(w)| = 2|\operatorname{Pin}(w)|$. Our goal in this section is to push this result a bit further and to completely characterize pinnacle sets.

Definition 2.1. A set S is an *n-admissible pinnacle set* if there exists a permutation $w \in S_n$ such that $\operatorname{Pin}(w) = S$. If S is n-admissible for some n, then we simply say that S is *admissible.*

Pinnacle sets are stable in the sense that if S is an n-admissible pinnacle set, then S is also $(n+1)$-admissible. Indeed, if $\operatorname{Pin}(w) = S$ for $w \in S_n$, $u = (n+1)w(1)\cdots w(n)$, and $v = w(1)\cdots w(n)(n+1)$, then

$$\operatorname{Pin}(u) = \operatorname{Pin}(v) = \operatorname{Pin}(w).$$

Moreover, any other way to insert $n+1$ into w will give a different pinnacle set, since $n+1$ would sit at a peak. Thus a kind of converse to this stability observation is the observation that if $\max S = m$, and S is an n-admissible pinnacle set for some $n \geq m$, then S is m-admissible. Extending this idea leads to the first half of the following result, characterizing admissible pinnacle sets.

Theorem 2.2 (Admissible pinnacle sets)**.** Let set S be a set of integers with $\max S = m$. Then S is an admissible pinnacle set if and only if both

1. $S \setminus \{m\}$ is an admissible pinnacle set, and
2. $m > 2|S|$.

Moreover, there are $\binom{m-2}{\lfloor m/2 \rfloor}$ admissible pinnacle sets with maximum m, and

$$1 + \sum_{m=3}^{n} \binom{m-2}{\lfloor m/2 \rfloor} = \binom{n-1}{\lfloor (n-1)/2 \rfloor},$$

admissible pinnacle sets $S \subseteq [n]$.

This characterization is in contrast to the characterization of peak sets. Whereas the number of peak sets is given by the Fibonacci numbers, here we get a central binomial coefficient. We now use this characterization of admissible pinnacle sets to count how many there are.

Definition 2.3. Given nonnegative integers m and d, define $\mathfrak{p}(m;d)$ to be the number of admissible pinnacle sets with maximum element m and cardinality d, using the convention $\mathfrak{p}(0;0) = 1$.

The following recurrence follows from the first half of Theorem 2.2:

$$\mathfrak{p}(m;d) = \begin{cases} 1 & \text{if } m = d = 0, \\ \sum\limits_{k<m} \mathfrak{p}(k;d-1) & \text{if } m > 2d, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The simplicity of the second half of Theorem 2.2 suggests a nice combinatorial explanation for the number of admissible pinnacle sets. Another nudge toward this combinatorial structure comes when we recognize that the numbers $\mathfrak{p}(m;d)$ satisfy a two-term recurrence for $m - 1 > 2d$:

$$\begin{aligned} \mathfrak{p}(m;d) &= \sum_{k<m} \mathfrak{p}(k;d-1), \\ &= \mathfrak{p}(m-1;d-1) + \sum_{k<m-1} \mathfrak{p}(k;d-1), \\ &= \mathfrak{p}(m-1;d-1) + \mathfrak{p}(m-1;d-1). \end{aligned}$$

The boundary cases for this recurrence are Catalan numbers, that is,

$$\mathfrak{p}(2d+1;d) = C_d$$

for $d \geq 1$, where $C_d = \binom{2d}{d}/(d+1)$. This hints at a connection between admissible pinnacle sets and lattice paths. We omit the discussion of this connection in the interest of space, but note here that this connection is critical in establishing the second half of Theorem 2.2.

3. Recurrences, explicit formulas, and bounds for $p_S(n)$

Now that we have characterized and enumerated admissible pinnacle sets, we turn to the question of counting permutations with a given pinnacle set.

To begin our study of $p_S(n)$, we make the easy observation that there are 2^{n-1} permutations in S_n having no peaks; that is, $p_\emptyset(n) = 2^{n-1}$. Indeed, if $\mathrm{Pin}(w) = \emptyset$, then we can write $w = u1v$, a concatenation of strings, where u is a word whose letters are strictly decreasing and v is a word whose letters are strictly increasing. If $w \in S_n$, then each such permutation is determined by the elements of u, which can be any subset of the $(n-1)$-element set $\{2, 3, \dots, n\}$. We will now analyze the number of ways to perform this procedure.

Definition 3.1. The *standardization map* relative to a set $X = \{x_1 < x_2 < \cdots\}$ is

$$\mathrm{std}_X(x_i) = i.$$

Fix a nonempty set $A = \{a_1 < a_2 < \cdots < a_{|A|}\} \subsetneq [n-1]$, and let

$$I = \mathrm{std}_A(S) = \{i : a_i \in S\}.$$

In other words, I is the set of relative values of pinnacles within the subset A.

With this notation, the number of permutations u of a set A such that $\mathrm{Pin}(u) = S'$ equals the number of permutations in $S_{|A|}$ with pinnacle set I. That is, the number of such u is $p_I(|A|)$. Likewise, letting $J = \mathrm{std}_{A^c}(S)$ denote the set of relative values of the pinnacles of within A^c, we have $p_J(|A^c|) = p_J(n-1-|A|)$ ways to form the permutation v.

Running over all cases of the set A, we get the following result.

Proposition 3.2 (A quadratic recurrence)**.** If S is an admissible pinnacle set with $\max S = n$, then

$$p_S(n) = \sum_{\emptyset \neq A \subsetneq [n-1]} p_{\mathrm{std}_A(S)}(|A|) \cdot p_{\mathrm{std}_{A^c}(S)}(n-1-|A|). \tag{1}$$

While it may seem that the quadratic recurrence must sum over $2^{n-1}-2$ subsets A, note that many of these selections contribute zero to the sum, because both $\mathrm{std}_A(S)$ and $\mathrm{std}_{A^c}(S)$ must themselves be admissible pinnacle sets. For example, with the set $S = \{4, 7, 9\}$, only 44 of the possible $2^8 - 2 = 254$ summands in Equation (1) are nonzero.

We can obtain explicit formulas for pinnacle sets with one or two elements.

Proposition 3.3. Let $3 \leq l < m$. For any $n \geq l$,

$$p_{\{l\}}(n) = 2^{n-2}(2^{l-2}-1)$$

and for any $n \geq m$,

$$p_{\{l,m\}}(n) = 2^{n+m-l-5}\left(3^{l-1} - 2^l + 1\right) - 2^{n-3}(2^{l-2}-1).$$

There may also be other special cases of explicit formulas that one can deduce from the quadratic recurrence, by exploring precisely which nonzero terms appear in the sum. For now, though, we turn to another recursive approach.

Proposition 3.4 (A linear recurrence)**.** Suppose that S is an admissible pinnacle set with $|S| = d$ and $\max S = m$. Then for any $n \geq m$,

$$p_S(n) = 2^{n-m}\left((m-2d)p_{S\setminus\{m\}}(m-1) + 2\sum_{\substack{T=(S\setminus\{m\})\cup\{j\}\\ j\in[m]\setminus S}} p_T(m-1)\right). \tag{2}$$

This linear recurrence tends to be very efficient in practice. It can also be used to yield explicit formulas when desired.

Proposition 3.5 (Enumerating permutations with maximal pinnacles)**.** Let d and n be any positive integers such that $2d < n$. The number of permutations in S_n with pinnacle set $[n-d+1, n] = \{n-d+1, n-d+2, \ldots, n\}$ is

$$p_{[n-d+1,n]}(n) = d! \cdot (d+1)! \cdot 2^{n-2d-1} \cdot S(n-d, d+1) \tag{3}$$

where $S(\cdot,\cdot)$ denotes the Stirling number of the second kind. Moreover, for any admissible pinnacle set $S \subseteq [n]$ with $|S| = d$, we have

$$p_S(n) \leq p_{[n+1-d,n]}(n).$$

It follows that a uniform upper bound over all S is achieved for the particular value of $d = |S|$ that maximizes the rightmost expression in Equation (3). While this choice of d appears to be a little less than $n/3$, we have no simple expression for d in terms of n.

Next we will state a result for minimizing $p_S(n)$ for a given size of admissible pinnacle set.

Proposition 3.6 (Enumerating permutations with minimal pinnacles)**.** Let d and n be any positive integers such that $2d < n$, and set

$$M_d := \{2k+1 : k = 1, \ldots, d\}.$$

Then the number of permutations in S_n with pinnacle set M_d is

$$p_{M_d}(n) = 2^{n-d-1}.$$

One could prove Proposition 3.6 by explicitly constructing such a permutation. For, if $w \in S_{2d+1}$ has $\mathrm{Pin}(w) = \{3, 5, \ldots, 2d+1\}$, then w has a simple structure: either $w = (2d)(2d+1)w'$ or $w = w'(2d+1)(2d)$, where w' has $\mathrm{Pin}(w') = M_{d-1}$. This choice of two options at each of d steps gives rise to 2^d such permutations.

If $w \in S_n$ has $\mathrm{Pin}(w) = M_d$, with $n > 2d+1$, then any numbers larger than $2d+1$ have the choice of going on the far left or far right of the permutation. That is, $w = uw'v$, where $w' \in S_{2d+1}$ has $\mathrm{Pin}(w') = M_d$, the elements of u are decreasing, and the elements of v are increasing. This structure helps to prove the following result.

Proposition 3.7 (Lower bounds)**.** Let d and n be any positive integers such that $2d < n$. Then for any admissible pinnacle set $S \subseteq [n]$ with $|S| = d$, we have

$$p_S(n) \geq p_{M_d}(n) = 2^{n-d-1}.$$

Putting this together with our previous propositions results in the following theorem.

Theorem 3.8 (Bounds on $p_S(n)$)**.** Let d and n be any positive integers such that $2d < n$. Then for any admissible pinnacle set $S \subseteq [n]$ such that $|S| = d$, we have the following sharp bounds:

$$2^{n-d-1} \leq p_S(n) \leq d! \cdot (d+1)! \cdot 2^{n-2d-1} \cdot S(n-d, d+1).$$

The results in this section allow us to find admissible pinnacle sets S that maximize and minimize $p_S(n)$, for fixed n. For the lower bound, we have

$$\min\{p_S(n) : \text{admissible } S \subseteq [n]\} = \min\{2^{n-d-1} : d < n/2\} = 2^{\lfloor n/2 \rfloor}.$$

For the upper bound, we have something a little less satisfying:

$$\max\{p_S(n) : \text{admissible } S \subseteq [n]\} = \max\{d!(d+1)!2^{n-2d-1}S(n-d, d+1) : d < n/2\}.$$

This introduces an interesting statistic.

Definition 3.9. For fixed n, let $d(n) = d < n/2$ be the value maximizing the expression $d!(d+1)!2^{n-2d-1}S(n-d, d+1)$.

Initially, this $d(n)$ appears to be a step function that increases by one as n increases by three. But $d(16) = 4$ shows that this is false. Data initially suggests that the step function cycles through seven plateaus of width three and an eighth plateau of width four, but this pattern also does not persist. For example, $d(n) = 12$ for the four consecutive values from $n = 38$ to $n = 41$ and $d(n) = 20$ for the four consecutive values from $n = 63$ to $n = 66$. But the next plateau of four is only seven steps away: $d(n) = 27$ from $n = 85$ to $n = 88$.

In Table 1 we list the values of n and $d(n)$ for which there are four consecutive values n with the same $d(n)$, i.e., for which $\{d(n), d(n+1), d(n+2), d(n+3)\}$ is a set of size 1. All other values of $d(n)$ that we have observed so far ($n \leq 200$) come in runs of three. The fact that the plateaus of size four are not quite periodic is puzzling. While it seems that $d(n)$ is approximately $n/3$, an exact formula for $d(n)$ (and hence the maximal value for $p_S(n)$) is so far elusive.

n	13	38	63	85	110	135	160	185
$d(n)$	4	12	20	27	35	43	51	59

TABLE 1. The values of $n \leq 200$ and corresponding $d(n)$ that mark the beginnings of four consecutive equal values: $d(n) = d(n+1) = d(n+2) = d(n+3)$.

References

[1] Sara Billey, Krzysztof Burdzy, and Bruce E. Sagan. Permutations with given peak set. *J. Integer Seq.*, 16(6):Article 13.6.1, 18, 2013.
[2] Richard Ehrenborg and Einar Steingrímsson. The excedance set of a permutation. *Adv. in Appl. Math.*, 24(3):284–299, 2000.
[3] Richard Ehrenborg and Einar Steingrímsson. Yet another triangle for the Genocchi numbers. *European J. Combin.*, 21(5):593–600, 2000.
[4] Dominique Foata and Doron Zeilberger. Denert's permutation statistic is indeed Euler-Mahonian. *Stud. Appl. Math.*, 83(1):31–59, 1990.
[5] Sergey Kitaev, Toufik Mansour, and Jeff Remmel. Counting descents, rises, and levels, with prescribed first element, in words. *Discrete Math. Theor. Comput. Sci.*, 10(3):1–22, 2008.
[6] J.-C. Novelli, J.-Y. Thibon, and L. K. Williams. Combinatorial Hopf algebras, noncommutative Hall-Littlewood functions, and permutation tableaux. *Adv. Math.*, 224(4):1311–1348, 2010.
[7] Einar Steingrímsson and Lauren K. Williams. Permutation tableaux and permutation patterns. *J. Combin. Theory Ser. A*, 114(2):211–234, 2007.

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