EQUIDISTRIBUTIONS OF MAHONIAN STATISTICS OVER PATTERN AVOIDING PERMUTATIONS (EXTENDED ABSTRACT)

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1. Introduction

A Mahonian d-function is a Mahonian statistic that can be expressed as a linear combination of vincular pattern statistics of length at most d. Babson and Steingrímsson [2] classified all Mahonian 3-functions up to trivial bijections (see Table 1) and identified many of them with well-known Mahonian statistics in the literature. In [1] we prove a host of Mahonian 3-function equidistributions over pattern avoiding sets of permutations. These are equidistributions of the form

$$\sum_{\sigma \in \mathrm{Av}_n(\pi_1)} q^{\mathrm{stat}_1(\sigma)} = \sum_{\sigma \in \mathrm{Av}_n(\pi_2)} q^{\mathrm{stat}_2(\sigma)} \tag{1.1}$$

where π_1, π_2 are permutation patterns and $\mathrm{stat}_1, \mathrm{stat}_2$ are (Mahonian) permutation statistics. When $\mathrm{stat}_1 = \mathrm{stat}_2$ such equidistributions coincide with the concept of *st-Wilf equivalence* coined by Sagan and Savage [15] and further studied in e.g. [7, 12, 8, 9, 5].

Although Mahonian statistics are equidistributed over $\mathcal{S}_n$, they need not be equidistributed over pattern avoiding sets of permutations. For instance maj and inv are not equidistributed over $\mathrm{Av}_n(\pi)$ for any classical pattern π of length three. Neither do the existing bijections in the literature for proving Mahonity necessarily restrict to bijections over $\mathrm{Av}_n(\pi)$. Therefore whenever such an equidistribution is present, we must usually seek a new bijection which simultaneously preserves statistic and pattern avoidance. Another motivation for studying equidistributions over $\mathrm{Av}_n(\pi)$ comes from the well-known fact that $|\mathrm{Av}_n(\pi)| = C_n$ for all $\pi \in \mathcal{S}_3$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. Therefore equidistributions of this kind induce equidistributions between statistics on other Catalan objects (and vice versa) whenever we have bijections where the statistics translate in an appropriate fashion. We prove several results in this vein where an exchange between statistics on $\mathrm{Av}_n(\pi)$, Dyck paths and polyominoes takes place. In general, studying the generating function (1.1) provides a rich source of interesting q-analogues to well-known sequences enumerated by pattern avoidance and raises new questions about the coefficients of such polynomials.

2. Main results

The equidistributions we mention below are shown using standard tools such as block decomposition, Dyck paths and generating functions. Our first result below may seem unexpected given that vincular patterns do not behave as straightforwardly under inverse as they do under complement and reverse.

Proposition 2.1. *Let $\sigma \in Av_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then*

$$\mathrm{mak}(\sigma) = \mathrm{imaj}(\sigma).$$

Name	Vincular pattern statistic	Reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [14]
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon [14]
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger [11]
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [6]
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [6]
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[2]
bast′	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[2]
bast″	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[2]
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [10]
foze′	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [10]
foze″	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [10]
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [15]
sist′	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [15]
sist″	$(\underline{13}2) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [15]

TABLE 1. Mahonian 3-functions.

Moreover for any $n \geq 1$,

$$\sum_{\sigma \in Av_n(\pi)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in Av_n(\pi^{-1})} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

Remark 2.2. By Proposition 2.1 and [17, Corollary 4.1] it follows that

$$\sum_{\sigma \in \mathrm{Av}_n(231)} q^{\mathrm{maj}(\sigma)+\mathrm{mak}(\sigma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \tag{2.1}$$

The right hand side of (2.1) is known as *MacMahon's q-analogue of the Catalan numbers.* In [7] it was moreover shown that $\sum_{\mathrm{Av}_n(231)} q^{\mathrm{inv}(\sigma)} = \tilde{C}_n(q)$ where $\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^k \tilde{C}_k(q) \tilde{C}_{n-k-1}(q)$ and $\tilde{C}_0(q) = 1$. The polynomial $\tilde{C}_n(q)$ is closely related to *Carlitz-Riordan's q-analogue of the Catalan numbers.*

The set of *descent bottoms* and *descent tops* of $\sigma \in \mathcal{S}_n$ are given by $\mathrm{DB}(\sigma) = \{\sigma(i+1) : i \in \mathrm{Des}(\sigma)\}$ and $\mathrm{DT}(\sigma) = \{\sigma(i) : i \in \mathrm{Des}(\sigma)\}$ respectively. Similarly the set of *ascent bottoms* and *ascent tops* of $\sigma \in \mathcal{S}_n$ are given by $\mathrm{AB}(\sigma) = \{\sigma(i) : i \in \mathrm{Asc}(\sigma)\}$ and $\mathrm{AT}(\sigma) = \{\sigma(i+1) : i \in \mathrm{Asc}(\sigma)\}$ respectively.

Theorem 2.3. *For any $n \geq 1$,*

$$\sum_{\sigma \in Av_n(321)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma \in Av_n(321)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)},$$
$$\sum_{\sigma \in Av_n(123)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)} = \sum_{\sigma \in Av_n(123)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)}.$$

The bijection used to prove Theorem 2.3 induces an interesting equidistribution on shortened polyominoes. A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

(i) P and Q begin at the same vertex and end at the same vertex.
(ii) P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$. For $(P,Q) \in \mathcal{H}_n$, let $\mathrm{Proj}_P^Q(i)$ denote the step $j \in [n]$ of P that is the *projection of the i^{th} step of Q on P*. Let

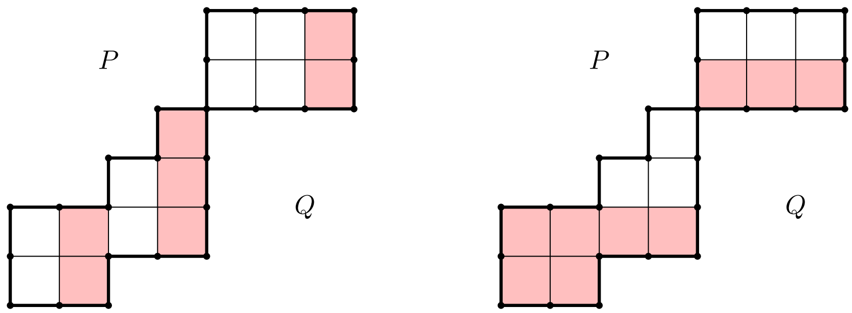
$$\mathrm{Valley}(Q) = \{i : Q_iQ_{i+1} = EN\}$$

denote the set of indices of the *valleys* in Q and let $\mathrm{nval}(Q) = |\,\mathrm{Valley}(Q)|$. Moreover for each $i \in [n]$ define

$$\mathrm{area}_{(P,Q)}(i) = \#\text{squares between the } i^{\text{th}} \text{ step of } Q \text{ and the } j^{\text{th}} \text{ step of } P,$$

where $j = \mathrm{Proj}_P^Q(i)$. Consider the statistics *valley-column area* and *valley-row area* of (P,Q) given by

$$\mathrm{vcarea}(P,Q) = \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i), \qquad \mathrm{vrarea}(P,Q) = \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i+1).$$



(a) $\mathrm{vcarea}(P,Q) = 2+3+2 = 7$ (b) $\mathrm{vrarea}(P,Q) = 2+4+3 = 9$

Theorem 2.4. *For any $n \geq 1$,*

$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vcarea}(P,Q)} t^{\mathrm{nval}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vrarea}(P,Q)} t^{\mathrm{nval}(Q)}.$$

Let $\mathrm{LRMin}(\sigma)$ denote the set of left-to-right minima in σ where $\sigma \in \mathcal{S}_n$.

Theorem 2.5. *For any $n \geq 1$,*

$$\sum_{\sigma\in Av(132)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)} = \sum_{\sigma\in Av(132)} q^{\mathrm{foze}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)}$$

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D enabling us to encode a Dyck path as a *Dyck word* (we will refer to the two notions interchangeably). Let $\mathcal{D}_n$ denote the set of all Dyck paths of length $2n$. For $P \in \mathcal{D}_n$, let $|P| = 2n$ denote the length of P. There are many statistics associated with Dyck paths in the literature. Here we will consider several Dyck path statistics that are intimately related with the inv statistic on pattern avoiding permutations.

Let $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$. A *double rise* in P is a subword UU and a *double fall* in P a subword DD. Let $\mathrm{dr}(P)$ and $\mathrm{df}(P)$ respectively denote the number of double rises and double falls in P. A *peak* in P is an up-step followed by a down-step, in other words, a subword of the form UD. Let $\mathrm{Peak}(P) = \{p : s_p s_{p+1} = UD\}$ denote the set of indices of the peaks in P and $\mathrm{npea}(P) = |\,\mathrm{Peak}(P)|$. For $p \in \mathrm{Peak}(P)$ define the *height* of p, $\mathrm{ht}_P(p)$, to be the y-coordinate of its highest point. A *valley* in P is a down step followed by an up step, in other words, a subword of the form DU. Let $\mathrm{Valley}(P) = \{v : s_v s_{v+1} = DU\}$ denote the set of indices of the valleys in P and $\mathrm{nval}(P) = |\mathrm{Valley}(P)|$. For $v \in \mathrm{Valley}(P)$ define the *height* of v, $\mathrm{ht}_P(v)$, to be the y-coordinate of its lowest point. For each $v \in \mathrm{Valley}(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = \mathrm{ht}_P(v)$ to the left of step v (see Figure 1). The length, $j - i$, of a tunnel is always an even number. Let $\mathrm{Tunnel}(P) = \{(i, j) : s_i \cdots s_j \text{ tunnel in } P\}$ denote the set of pairs of beginning and end indices of the tunnels in P. Cheng et.al. [4] define the statistics *sumpeaks* and *sumtunnels* given respectively by

$$\mathrm{spea}(P) = \sum_{p \in \mathrm{Peak}(P)} (\mathrm{ht}_P(p) - 1), \qquad \mathrm{stun}(P) = \sum_{(i,j) \in \mathrm{Tunnel}(P)} (j - i)/2.$$

Let $\mathrm{Up}(P) = \{i : s_i = U\}$ denote the indices of the set of U-steps in P. Given $i \in \mathrm{Up}(P)$ define the *height* of i, $\mathrm{ht}_P(i)$, to be the y-coordinate of its leftmost point. Define the statistic *sumups* by

$$\mathrm{sups}(P) = \sum_{i \in \mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil.$$

Burstein and Elizalde [3] define a statistic which they call the mass of P. For each $i \in \mathrm{Up}(P)$ define the *mass* of i, $\mathrm{mass}_P(i)$, as follows. If $s_{i+1} = D$, then $\mathrm{mass}_P(i) = 0$. If $s_{i+1} = U$, then P has a subword of the form $s_i U P_1 D P_2 D$ where P_1, P_2 are Dyck paths and we define $\mathrm{mass}_P(i) = |P_2|/2$. In other words, the mass is half the number of steps between the matching D-steps of two consecutive U-steps. Define

$$\mathrm{mass}(P) = \sum_{i \in \mathrm{Up}(P)} \mathrm{mass}_P(i).$$

The *standard bijection* $\Delta : \mathrm{Av}_n(231) \to \mathcal{D}_n$ can be defined recursively by $\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2)$, where

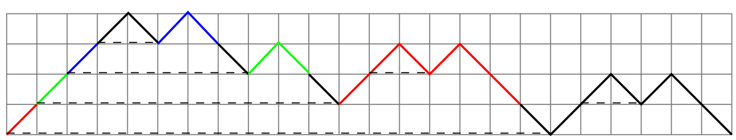


FIGURE 1. The tunnel lengths of a Dyck path (indicated with dashes) and the mass associated with the first three up-steps is highlighted with matching colours.

$\sigma = 213[1, \sigma_1, \sigma_2]$. The following theorem is essentially a restatement of [3, Theorem 3.11]

Theorem 2.6. *For all $\sigma \in Av_n(231)$ and $P \in \mathcal{D}_n$ we have*

(i) $\mathrm{mad}(\sigma) = \mathrm{mass}(\Delta(\sigma)) + \mathrm{dr}(\Delta(\sigma))$,

(ii) a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ such that $\mathrm{sups}(P) = \mathrm{mass}(\Theta(P)) + \mathrm{dr}(\Theta(P))$.

Theorem 2.7. *There exists a bijection $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ such that* $\mathrm{stun}(P) = \mathrm{mass}(\Phi(P)) + \mathrm{dr}(\Phi(P))$. *In particular for any $n \geq 1$,*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{mass}(P) + \mathrm{dr}(P)}.$$

Corollary 2.8. *For any $n \geq 1$,*

$$\sum_{\sigma \in Av_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in Av_n(321)} q^{\mathrm{inv}(\sigma)}.$$

Proof. By [4, Proposition 4.1], [4, Theorem 4.2], Theorem 2.7 and Theorem 2.6 (i) we have the following diagram of weight preserving bijections

$$\begin{tikzcd}
(\mathrm{Av}_n(321), \mathrm{inv}) \arrow[r, "\Gamma"] \arrow[d, "\phi"] & (\mathcal{D}_n, \mathrm{spea}) \arrow[r, "\Psi"] & (\mathcal{D}_n, \mathrm{stun}) \arrow[d, "\Phi"] \\
(\mathrm{Av}_n(231), \mathrm{mad}) \arrow[rr, "\Delta"] & & (\mathcal{D}_n, \mathrm{mass} + \mathrm{dr})
\end{tikzcd}$$

Thus $\phi = \Delta^{-1} \circ \Phi \circ \Psi \circ \Gamma$ is our sought bijection with $\mathrm{inv}(\sigma) = \mathrm{mad}(\phi(\sigma))$ for all $\sigma \in \mathrm{Av}_n(321)$.

□

The following corollary answers a question of Burstein and Elizalde in [3].

Corollary 2.9. *There exists a bijection* $\Lambda : \mathcal{D}_n \to \mathcal{D}_n$ *such that* $\mathrm{spea}(P) = \mathrm{sups}(\Lambda(P))$*. In particular for any* $n \geq 1$*,*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

A complete summary of proved (and conjectured) equidistributions may be found in [1, Table 2].

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