Rare and Not-So-Rare Regions of Pattern-Avoiding Permutations

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For a given pattern τ of length k, our goal is to describe some probabilistic features of the random permutation σ drawn uniformly from $\mathrm{AV}_n(\tau)$, the set of τ-avoiding permutations of length n, where n is large.

An open subset B of the unit square $[0,1]^2$ is called *τ-rare* if it is exponentially unlikely to contain any point of the scaled-down plot of σ. That is, writing P_n^τ for the uniform probability measure on $\mathrm{AV}_n(\tau)$, we say that B is rare if there exists a positive constant c such that

$$P_n^\tau\left(\left\{\left(\frac{i}{n},\frac{\sigma(i)}{n}\right):\ i=1,\ldots,n\right\}\cap B\neq\emptyset\right)\ \leq\ e^{-cn} \qquad (1)$$

for all sufficiently large n. The *rare region* for the pattern τ is the union of all the τ-rare open sets. As illustrated in Figure 1, simulations suggest that the rare region of the pattern **4231** has components above and below the diagonal $y=x$, and that the rare region of **213** consists only of the triangle above the diagonal.

The first part of this talk will describe what we know and what we conjecture about rare regions. A basic result is the following.

Theorem [1] *Assume without loss of generality that $\tau_1>\tau_k$. Then the part of the rare region above the diagonal is nonempty if and only if $\tau_1=k$.*

If τ is the decreasing permutation $\mathbf{k(k{-}1)\cdots 321}$, then it is not hard to see that the rare region is everything off the diagonal; more interestingly, we can compute the exact exponential decay rates of the left-hand side of Equation (1) [3].

For a general pattern with $\tau_1=k$, there is a continuous curve from $(0,0)$ to $(1,1)$ that is the boundary of the part of the rare region lying above the diagonal [4]. We do not know whether this curve can lie strictly above the diagonal. However, we do know some properties of this curve, including the fact that is strictly increasing with slope bounded by $1/L(\tau)$, where $L(\tau)$ is the Stanley-Wilf limit of τ. We do not know whether this curve must be concave.

The second part of the talk concerns patterns with empty rare regions. Here we can ask about the (subexponential) rate of decay of the probability that the plot includes certain points, i.e.

$$P_n^\tau(\sigma(i_n)=j_n)\sim ?? \quad \text{when } \lim_{n\to\infty}\left(\tfrac{i_n}{n},\tfrac{j_n}{n}\right)\ =\ (x,y)\ \in[0,1]^2.$$

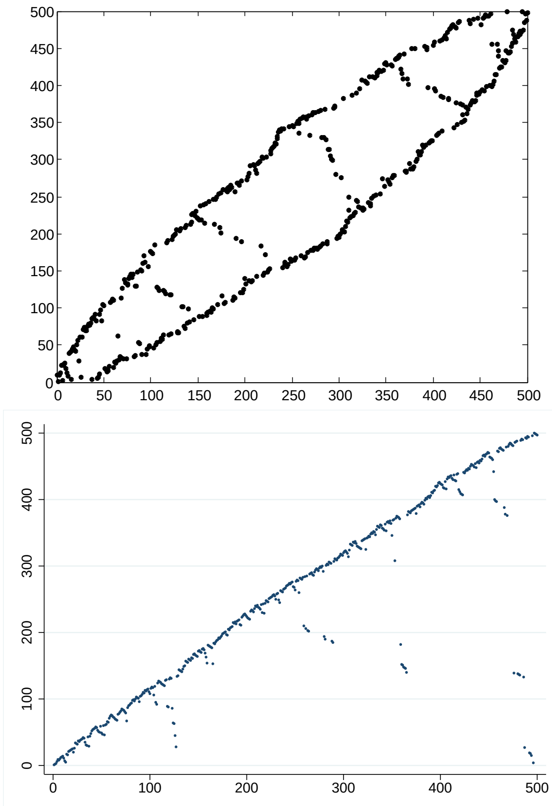

Figure 1: A random **4231**-avoiding permutation (top) and a random **312**-avoiding permutation (bottom). Both have $N = 500$.

For the case $\tau = \mathbf{312}$ and $0 < y < x < 1$ (that is, interior to the triangle below the diagonal), we can evaluate the probability exactly, and it is asymptotically proportional to $n^{-3/2}$ [5, 2]. A more challenging case is $\tau = \mathbf{2413}$, where it seems harder to obtain exact probabilities. I will present some preliminary work on this with Gökhan Yıldırım.

References

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