The first order logic of permutations

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The set of all finite permutations $\mathcal{S}$ can be thought of as containing a single representative of each isomorphism class of the set of all finite models of the theory of two linear orders (TOTO). This viewpoint of permutations is really the *permutation diagram* perspective, and in this context the substructures of a given model correspond exactly to the subpermutations of the corresponding permutation in the classical sense.

However, permutations are of course also naturally considered as bijective functions from a set to itself and it seems natural to ask – what properties of these functions (if any) are expressible in TOTO?

The initial indications are not promising:

Proposition 1. *There is no formula* $\mathrm{FP}(x)$ *in TOTO such that for any permutation* π *and any* $a \in \pi$, $\pi \models \mathrm{FP}(a)$ *if and only if* a *is a fixed point of* π*, nor is there a sentence* FP *of TOTO such that for any permutation* π, $\pi \models \mathrm{FP}$ *if and only if* π *has a fixed point.*

In light of this negative result we restrict our attention to classical permutation classes and ask: for which classes $\mathcal{C}$ is the positive version of the preceding result true? Let I be the class of all increasing permutations, D the class of all decreasing permutations, and $\mathbf{1}$ the class consisting of the single permutation 1.

It turns out not to matter whether we are looking for the definability of fixed points by formula, or of their existence by a sentence.

Proposition 2. *Let* $\mathcal{C}$ *be a permutation class. The following are equivalent:*

- *There is a formula* $\mathrm{FP}_{\mathcal{C}}(x)$ *which defines fixed points in permutations belonging to* $\mathcal{C}$ *(i.e., for* $a \in \pi \in \mathcal{C}$, $\pi \models \mathrm{FP}_{\mathcal{C}}(a)$ *if and only if* a *is a fixed point of* π*).*
- *There is a sentence* $\mathrm{FP}_{\mathcal{C}}$ *which characterises the existence of fixed points in permutations belonging to* $\mathcal{C}$ *(i.e., for* $\pi \in \mathcal{C}$, $\pi \models \mathrm{FP}_{\mathcal{C}}$ *if and only if* π *has a fixed point.)*
- $\mathcal{C}$ *avoids at least one permutation in* D *and at least one permutation in* $\mathsf{I} \ominus \mathbf{1} \ominus \mathsf{I}$.

Emboldened by this success we consider the obvious generalisation of the preceding results. For $\sigma \in \mathcal{S}_k$ and $\pi \in \mathcal{S}_n$, we say that σ is a *stable subpermutation* of π if there is some k-element subset $\Sigma \subseteq [n]$ such that π maps Σ to Σ, and the pattern of π on Σ equals σ. We call the set Σ (or a sequence consisting of its elements) a *stable occurrence* of σ in π. So for example the stable occurrences of 1 in π are just its fixed points and the stable

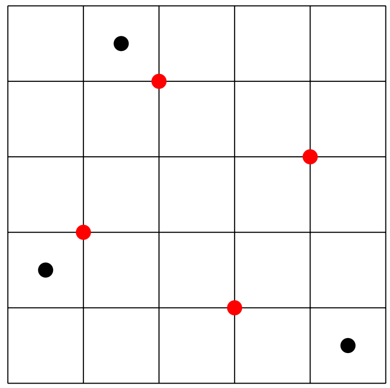

Figure 1: The dots show the permutation $E(2413, (1, 2, 5)) = 3476251$.

subpermutations that π contains are just the subpermutations defined on unions of the cycles of π.

For $\sigma \in \mathcal{S}_k$ and $\mathbf{i}$ a sequence of r distinct elements from $[k+1]$ we define the *expansion of* σ *by* $\mathbf{i}$, $E(\sigma, \mathbf{i})$ to be that permutation of length $k+r$ containing a stable occurrence, $\mathbf{s}$ of σ and for which the remaining r elements define the cycle $\mathbf{i}$ in the $(k+1) \times (k+1)$ grid defined by $\mathbf{s}$ (see Figure 1). Given a sequence of r permutations $\Theta = (\theta_1, \theta_2, \ldots, \theta_r)$ we define the *inflation of* $E(\sigma, \mathbf{i})$ *by* Θ *on* $\mathbf{i}$ to be the permutation, $E(\sigma, \mathbf{i}, \Theta)$, obtained by inflating those points of $E(\sigma, \mathbf{i})$ corresponding to the elements of $\mathbf{i}$ in left to right order by the permutations θ_1, θ_2, etc. Similarly, if $\overline{\mathcal{X}} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_r)$ is a sequence of permutation classes, then we define the *inflation of* $E(\sigma, \mathbf{i})$ *by* $\overline{\mathcal{X}}$ *on* $\mathbf{i}$ to be the permutation class, $E(\sigma, \mathbf{i}, \overline{\mathcal{X}})$, consisting of all the subpermutations of permutations in $E(\sigma, \mathbf{i}, \Theta)$ where for $1 \leq t \leq r$, $\theta_t \in \mathcal{X}_t$.

Now we obtain:

Theorem 3. *Let $\sigma \in \mathcal{S}_k$ be a permutation, and let $\mathcal{C}$ be a permutation class. The following conditions are equivalent:*

- *there is a formula $\psi(\mathbf{x})$ of TOTO such that for $\pi \in \mathcal{C}$ and sequences $\mathbf{s}$ from π, $\pi \models \psi(\mathbf{s})$ if and only if $\mathbf{s}$ is a stable occurrence of σ in π; and*
- *for each non-trivial cycle $\mathbf{i}$ of distinct elements from $[k+1]$ and each sequence $\overline{\mathcal{X}}$ consisting of Is and Ds of the same length as $\mathbf{i}$ the class $\mathcal{C}$ does not contain $E(\sigma, \mathbf{i}, \overline{\mathcal{X}})$, i.e., it avoids at least one permutation in each such class.*

The proofs of these results make use of a "magic lemma" related to Ehrenfeucht-Fraïssé, or Duplicator-Spoiler games.