Pattern Avoiding Linear Extensions of Rectangular Posets

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1 Introduction

In [Yak15] Yakoubov introduced an extensive new family of permutation enumeration problems. To state the most general of these problems, suppose n is a positive integer, $\Uparrow$ is a partial order on $[n]$, and $\sigma_1, \ldots, \sigma_k$ are permutations. Then Yakoubov's problem is to determine how many permutations π of $[n]$ avoid $\sigma_1, \ldots, \sigma_k$ and also have the property that if $\pi(i) \Uparrow \pi(j)$ then $i < j$. In other words, how many linear extensions of the poset $([n], \Uparrow)$ avoid $\sigma_1, \ldots, \sigma_k$?

As Yakoubov points out, this problem is hopelessly general without some additional information about $\Uparrow$. For example, if $\Uparrow$ is empty (meaning no two elements of $[n]$ are related by $\Uparrow$) then Yakoubov's question reduces to the problem of enumerating the permutations avoiding a given set of patterns, a problem about which much has been written over the past thirty years, and about which much more is still unknown. On the other hand, as Yakoubov also illustrates, for some particular families of partial orders we can make significant progress for a variety of short forbidden patterns. In particular, Yakoubov obtains simple closed formulas for the number of linear extensions of posets she calls combs (see Figure 1 for two typical examples) which avoid various sets of patterns of length three.

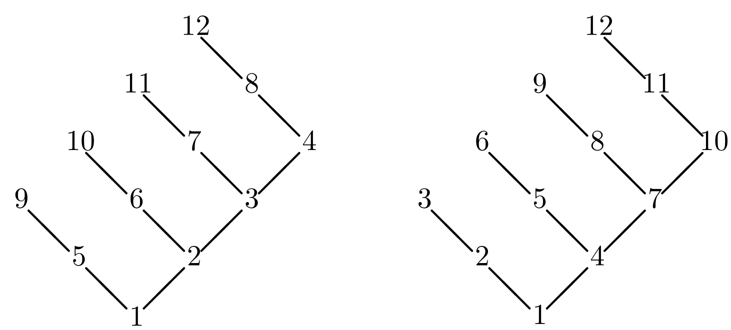

Figure 1: The Hasse diagrams of a comb of type α (left) and β (right).

In this work we extend Yakoubov's investigation by studying pattern avoiding linear extensions of rectangular posets (see Figure 2 for a typical example). We will first give some background on pattern avoidance and partially ordered sets, we define the particular posets we plan to study, and we mention some preliminary results to reduce the scope of our problem.

2 Background and Notation

Our language and notation for partially ordered sets, their Hasse diagrams, and their linear extensions will follow [Sta11, Chap. 3]. Because Hasse diagrams are often the clearest way to describe the relations in a poset, we sometimes use them to define our posets.

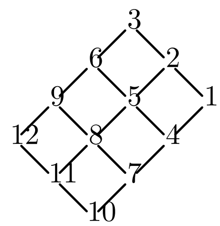

Figure 2: The Hasse diagram of the rectangular poset $EN_{4,3}$.

For any poset $P = (X, \Uparrow)$, a *linear extension* of P is a total ordering $\prec$ of X which is consistent with $\Uparrow$. In other words, if $a \Uparrow b$ then $a \prec b$. When $X = [n]$ for some positive integer n, these linear extensions are naturally associated with permutations in S_n. Specifically, the permutation π associated with $\prec$ is the permutation with $\pi(1) \prec \pi(2) \prec \cdots \prec \pi(n)$. For example, if P is the poset whose Hasse diagram is given in Figure 3, then P has four linear extensions: 25413, 25431, 52413, and 52431.

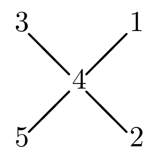

Figure 3: The Hasse diagram of a poset on [5].

Many of our posets will have the property that their Hasse diagrams can be represented as tilted rectangles, as in Figure 2; we call such a poset *rectangular*. Following Yakoubov, for a given rectangular representation of a rectangular poset, we call the sequences of elements in the diagonals from lower left to upper right the *spines*, and we write s to denote the number of elements in each spine. Similarly, we call the sequences of elements in the diagonals from lower right to upper left the *teeth*, and we write t to denote the number of elements in each tooth. We number the spines from upper left to lower right, and the teeth from upper right to lower left. In Figure 2 we have $s = 4$, $t = 3$, the first spine contains 3, 6, 9, and 12, and the third tooth contains 7, 8, and 9.

Given positive integers s and t, there are eight natural rectangular partial orderings of $[st]$ with spine length s and tooth length t. For example, in one natural rectangular partial ordering the first spine is $1, 2, \ldots, s$ from first tooth to last, the second spine is $s+1, s+2, \ldots, 2s$ from first tooth to last, and in general the jth spine is $(j-1)s+1, (j-1)s+2, \ldots, js$ from first tooth to last. In another natural rectangular partial ordering the sth tooth is $t, t-1, \ldots, 1$ from first spine to last, the $(s-1)$th tooth is $2t, 2t-1, \ldots, t+1$ from first spine to last, and in general the $(s-j)$th tooth is $(j+1)t, (j+1)t-1, \ldots, jt+1$ from first spine to last. In each of these eight natural rectangular partial orderings one spine or tooth is $1, 2, \ldots$ in order, so we name each ordering according to the corners of the Hasse diagram (North, East, South, or West) at which this spine or tooth begins and ends. For example, in Figure 2 we have the Hasse diagram for $EN_{4,3}$, and in Figure 4 we have the Hasse diagram for $SW_{2,4}$. Finally, for any rectangular poset $P = ([st], \Uparrow)$ and any permutations $\sigma_1, \ldots, \sigma_n$, we write $P(\sigma_1, \ldots, \sigma_n)$ to denote the set of linear extensions of P which avoid each permutation $\sigma_1, \ldots, \sigma_n$.

Although there are eight natural rectangular partial orderings on $[st]$, when counting their pattern avoiding linear extensions we need only concern ourselves with two of them. For instance, by reflecting their Hasse diagrams over a vertical line, we can see that $EN_{s,t} = WN_{t,s}$, $NE_{s,t} = NW_{t,s}$, $ES_{s,t} = WS_{t,s}$, and $SE_{s,t} = SW_{t,s}$. Similarly, reflecting a Hasse diagram over a horizontal line has the effect of reversing the linear extensions of the associated poset. As a result of these observations,

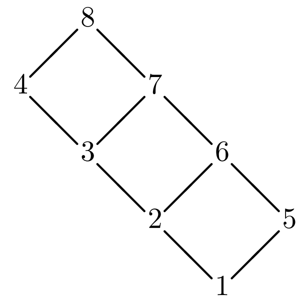

Figure 4: The Hasse diagram of the rectangular poset $SW_{2,4}$.

we see we can restrict our attention to $EN_{s,t}(\sigma_1, \ldots, \sigma_n)$ and $NE_{s,t}(\sigma_1, \ldots, \sigma_n)$.

In addition to reducing the collection of posets we need to consider, we can also reduce the collection of forbidden patterns we need to consider using the standard reverse complement map.

We now have all of the background and notation we need to begin counting pattern avoiding linear extensions of $EN_{s,t}$ and $NE_{s,t}$. Before we do this, it's natural to ask what happens when there are no patterns to avoid. That is, how many linear extensions of $EN_{s,t}$ and $NE_{s,t}$ are there?

The classical hook length formula [GNW79] [Sag01, Sec. 3.10] for the number of standard tableaux of an arbitrary partition shape gives us the following result.

Proposition 2.1. *For any positive integers s and t, the number of linear extensions of $EN_{s,t}$ (or $NE_{s,t}$) is* $(st)! \prod_{j=1}^{t} \frac{(s+t-j)!}{(j-1)!}$.

Since the Catalan numbers are so ubiquitous, it's worth noting their appearance in a special case of Proposition 2.1.

Corollary 2.2. *For any positive integer n, the number of linear extensions of any of $EN_{n,2}$, $NE_{n,2}$ $EN_{2,n}$, or $NE_{2,n}$ is the Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

3 Results

Although we considered linear extensions avoiding sets of patterns of length three, finding simple closed formulas in most cases, we omit those results here for space. To begin our investigation of patterns of length 4, we use a generating tree to show that the Fuss-Catalan numbers enumerate the linear extensions of the EN poset which avoid 1243.

Theorem 3.1. *For all $s, t \geq 1$,*

$$|EN_{s,t}(1243)| = \frac{1}{(t-1)s+1}\binom{st}{s}.$$

We then give a natural bijection (which we omit here for space) between this set of linear extensions and the set of Fuss-Catalan paths. We also use the transfer matrix method, again omitted for space, to enumerate a set of linear extensions avoiding 2143. Finally, we extend Yakoubov's work in another direction, studying the distribution of the inversion number (using q in the standard way) on pattern avoiding linear extensions of our rectangular posets.

Theorem 3.2. *For all $s \geq 1$ and all $t \geq 1$,*

$$NE_{s,t}(213)(q) = q^{s\binom{t}{2}+t\binom{s}{2}+\frac{(s-1)(t-1)(st-2)}{2}}[t]_q^{s-1}. \tag{1}$$

Returning now to $EN_{s,t}(1243)$, we give the range of possible inversion numbers.

Theorem 3.3. *For any positive integers s and t, the inversion numbers of the linear extensions of $EN_{s,t}$ which avoid 1243 have minimum $(t^2 - t + 1)\binom{s}{2}$ and maximum $t^2\binom{s}{2}$.*

Proof. To show the minimum number of inversions is $(t^2 - t + 1)\binom{s}{2}$, first note that by a theorem in the complete paper, each entry which is not on the tth tooth must form an inversion with all entries on higher (that is, lower-numbered) spines. In particular, each entry on the jth tooth that is not on the tth spine contributes $(j - 1)(t - 1)$ inversions, so the jth tooth contributes $(j-1)(t-1)^2$ inversions. Now note that each entry on the nth tooth forms an inversion with all of the entries on higher (that is, lower-numbered) teeth, so the jth entry on the tth tooth contributes $(j - 1)t$ inversions. In total, then, each linear extension of $EN_{s,t}$ which avoids 1243 has at least $\sum_{j=1}^{s}((j - 1)(t - 1)^2 + (j - 1)t) = \binom{s}{2}(t^2 - t + 1)$ inversions. Since the linear extension which begins with the entries on the sth spine has no other inversions, it must have exactly $\binom{s}{2}(t^2-t+1)$ inversions.

To show the maximum number of inversions is $t^2\binom{s}{2}$, for each linear extension π we consider pairs $i < j$ with $\pi(i) < \pi(j)$. That is, we consider the noninversions or coinversions in π. On each tooth there are $\binom{t}{2}$ such pairs, so no linear extension has more than $\binom{st}{2} - s\binom{t}{2} = t^2\binom{s}{2}$ inversions. Since the linear extension we obtain by traversing the teeth from sth to first has no other noninversions, it must have exactly $t^2\binom{s}{2}$ inversions. □

Theorem 3.3 captures a striking observation about the data in Table 1, which shows the minimum and maximum inversion numbers among linear extensions in $EN_{s,t}(1243)$ for various s and t. Namely, for linear extensions in $EN_{s,t}(1243)$, the minimum inversion number is the product of the inversion number of the linear extension in $EN_{s,1}(1243)$ and the minimum inversion number over linear extensions in $EN_{2,t}(1243)$. Similarly, the maximum inversion number is the product of the inversion number of the linear extension in $EN_{s,1}(1243)$ and the maximum inversion number over linear extensions in $EN_{2,t}(1243)$. In other words, in Table 1 the entry in row i and column j is the product of the entry in row i and column 1 with the entry in row 2 and column j.

$s \setminus t$	1	2	3	4	5	6
1	0	0	0	0	0	0
2	1	3-4	7-9	13-16	21-25	31-36
3	3	9-12	21-27	39-48	63-75	93-108
4	6	18-24	42-54	78-96	126-150	196-216
5	10	30-40	70-90	130-160	210-250	310-360
6	15	45-60	105-135	195-240	315-375	465-540

Table 1: The range of inversion numbers (minimum-maximum) for linear extensions of $EN_{s,t}(1243)$.

4 Conjectures

We have found that the following conjecture holds for $t \leq 9$.

Conjecture 4.1. *For all $t \geq 1$,*

$$EN_{3,2t-1}(1243)(q) = q^{3(t^2-t+1)}[2t-1]_q[4t-1]_q.$$

Our next conjecture, which we have verified for $s \leq 9$, is a q-analogue of a theorem from an earlier section which was omitted for space.

Conjecture 4.2. *For all $s \geq 1$,*

$$EN_{s,3}(2143)(q) = q^{9\binom{s}{2}} f_s\left(\frac{1}{q}\right),$$

where $f_s(q)$ is defined by $f_0(q) = 1$, $f_1(q) = 1$, and $f_s(q) = (1 + q + 2q^2)f_{s-1}(q) + q^3 f_{s-2}(q)$ for $s \geq 2$.

The complete paper (of which this abstract contains excerpts) has been accepted to and will appear in the *Journal of Combinatorics* and is also available on arXiv.org.

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