Sorting with Pop Stacks

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A *stack* is a last-in first-out data structure with two operations:

- *push* – remove the first element from input and put it on the top of the stack
- *pop* – remove the top element from the stack and put it on the end of the output.

When we sort a permutation with a stack, the usual convention is to keep the stack elements in increasing order so that the stack is not used to create inversions in the output permutation that were not present in the input permutation. If we sort using an increasing stack, the stack sorting operation $S(\pi)$ is well-defined. In particular, for $\pi \in \mathcal{S}_n$, suppose that $\pi_i = n$. Let $L = \pi_1 \cdots \pi_{i-1}$ and let $R = \pi_{i+1} \cdots \pi_n$. Then $S(\pi) = S(L)S(R)n$, where $S(\pi) = \pi$ if $|\pi| \leq 1$. Knuth [4] showed that a permutation π is sortable after one pass through a stack (i.e. $S(\pi) = 12 \cdots n$) if and only if π avoids the pattern 231, and it is well-known that there are Catalan-many such permutations.

In 1990, West [5] considered permutations that are sortable after two passes through a stack, i.e. permutations where $S(S(\pi)) = 12 \cdots n$. This is different from sorting with stacks in series or in parallel. West showed that $S(S(\pi)) = 12 \cdots n$ if and only if π avoids 2341 and $3\bar{5}241$. Two years later, Zeilberger [6] showed that there are $\dfrac{2(3n)!}{(n+1)!(2n+1)!}$ such permutations.

A pop stack is a modified stack. The push operation is the same as for a conventional stack. However, whenever the pop operation occurs, now we must output *all* elements from the pop stack at once. If we maintain the convention of keeping pop stack elements in increasing order, the pop stack operation $P(\pi)$ is also well-defined. In particular, for $\pi \in \mathcal{S}_n$, suppose that $\pi_1 \cdots \pi_i$ is the longest decreasing prefix of π. Then $P(\pi) = \pi_i \cdots \pi_1 P(\pi_{i+1} \cdots \pi_n)$ where $P(\pi) = \pi$ if $|\pi| \leq 1$. Avis and Newborn [3] studied pop stacks in series. They proved that $P(\pi) = 12 \cdots n$ if and only if π avoids 231 and 312 and there are 2^{n-1} such permutations of length n. In fact, $P(\pi) = 12 \cdots n$ if and only if π is a layered permutation. Later, Atkinson and Sack [2] studied pop stacks in parallel. However, the pop stack analogue of West 2-stack-sortable permutations has been left open until now.

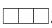
In this talk, we characterize permutations such that $P(P(\pi)) = 12 \cdots n$. We say such permutations are 2-pop-stack-sortable. In particular, we show the following:

Theorem 1. *Consider $\pi \in \mathcal{S}_n$. $P(P(\pi)) = 12 \cdots n$ if and only if π avoids 2341, 3412, 3421, 4123, 4231, 4312,* $4\bar{1}352$*, and* $413\bar{5}2$.

We also enumerate these permutations.

Theorem 2. *Let $P_n = \{\pi \in \mathcal{S}_n | P(P(\pi)) = 12 \cdots n\}$. Then:*

$$\sum_{n \geq 1} |P_n| \, x^n = \frac{x(x^2+1)}{1 - 2x - x^2 - 2x^3}.$$

Both 1-pop-stack-sortable and 2-pop-stack-sortable permutations are in bijection with certain families of polyominoes. Here, a polyomino is a collection of n equal squares joined with coincident sides. For example, the six polyominoes of size 3 are shown in Figure 1. Instead of conventional polyominoes, Aleksandrowich, Asinowski, and Barequet [1] enumerated polyominoes on a twisted cylinder. For example, a flattened version of a twisted cylinder of width 2 is shown in Figure 2. In a twisted cylinder, the rightmost square on one row of the cylinder is adjacent to the leftmost square on the next row of the cylinder. With this convention, there are only four polyominoes of size 3 on a twisted cylinder of width 2; these are shown in Figure 3. Notice that , , and  are all the same polyomino now on the twisted cylinder of width 2. We show that 1-pop-stack-sortable permutations are in bijection with polyominoes on a twisted cylinder of width 2 and 2-pop-stack-sortable permutations are in bijection with polyominoes on a twisted cylinder of width 3.

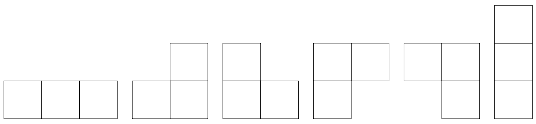


Figure 1: The six polyominoes of size 3

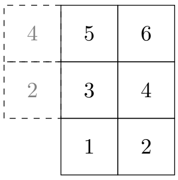


Figure 2: A twisted cylinder of width 2

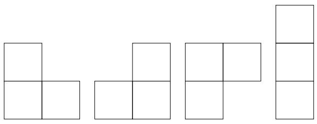


Figure 3: The four polyominoes of size 3 on a twisted cylinder of size 2

This talk is based on joint work with Rebecca Smith.

References

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