Quarter Turn Baxter Permutations

Kevin Dilks

May 29, 2017

Abstract

Baxter permutations are known to be in bijection with a wide number of combinatorial objects. Previously, it was shown that each of these objects had a natural involution which was carried equivariantly by the known bijections, and the number of objects fixed under involution was given by Stembridge's "q=-1" phenomenon. In this paper, we consider the order 4 action of a quarter-turn rotation of a Baxter permutation matrix, refining the half-turn rotation previously studied. Using the method of generating trees, we show that the number of Baxter permutations fixed under quarter-turn rotation has a very nice enumeration, which suggests the existence of a combinatorial bijection.

Keywords: Baxter permutations, symmetry, pattern avoidance, generating trees

1 Baxter Permutations

Baxter permutations are a well-studied class of permutations, which have a number of symmetries and nice properties associated to them.

Definition 1.1. We say that a *Baxter permutation* is a permutation that avoids the patterns 3-14-2 and 2-41-3, where an occurrence of the pattern 3-14-2 in a permutation $w = w_1 \dots w_n$ means there exists a quadruple of indices $\{i, j, j+1, k\}$ with $i < j < j+1 < k$ and $w_j < w_k < w_i < w_{j+1}$ (and similarly for 2-14-3)[1].

It is easy to see from the definition that Baxter permutations will be closed under two natural involutions. One of them reverses the order of a word ($w = w_1 \dots w_n \mapsto w_n \dots w_1$), and the other reverses the order of the labels ($w = w_1 \dots w_n \mapsto (n+1-w_1) \dots (n+1-w_n)$). These correspond to reflecting a permutation matrix horizontally and vertically (respectively). It is slighly less clear that Baxter permutations will be closed under taking inverses, which corresponds to reflecting the permutation matrix across a diagonal line. This means that Baxter permutations carry the full dihedral action of the square.

[1]Such patterns are sometimes called *vincular* patterns.

It is clear that the first two involutions individually will never have any fixed points for $n > 1$.

The author has previously shown that the combination of the first two involutions (correspond to a half-turn of the permutation matrix) is carried equivariantly to a natural rotation on other combinatorial objects, and the enumeration of fixed points is an instance of the "$q = -1$ phenomenon". [2]

Baxter permutations fixed under reflection across the diagonal correspond to self-involutive Baxter permutations, and these have previously been considered. Bousqet-Mélou came up with enumerative formulas for the number of fixed-point free self-involutive Baxter permutations of length $2n$, which has the surprisingly simple closed formula $b_n = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$, as well as refined enumeration with respect to various statistics [1]. Later, Fusy used planar maps to give a combinatorial proof of the enumeration for b_n, as well as a closed-form multivariate enumeration for all self-involutive Baxter permutations [1].

The last remaining conjugacy class of dihedral actions on Baxter permutations is the one corresponding to $90°$ rotations, which we now consider.

Theorem 1.2. *The number of Baxter permutations of length n fixed under $90°$ rotation of its permutation matrix is $2^m C_m$ (where C_m is the Catalan number) if $n = 4m + 1$, and zero otherwise.*

To prove this, we will recall and extend the method of generating tress originally used by Chung, Graham, Hoggat, and Kleiman to enumerate all Baxter permutations.

1.1 Generating Tree for Baxter permutations

Baxter permutations are given by a vincular pattern, where we have adjacency issues to consider, so it is not immediately obvious that they are closed under removing the largest label.

Lemma 1.3. *If w is a Baxter permutation, and we remove its largest label, then the result is still a Baxter permutation*

Therefore, every Baxter permutation of length n uniquely arises from taking a Baxter permutation of length $n-1$ and inserting n into an admissible position.

The admissible places where we can insert a new largest label into a Baxter permutation are immediately to the left of a left-to-right maxima, and immediately to the right of a left-to-right maxima.

The resulting Baxter permutation will also have a predictable number of left-to-right and right-to-left maxima. Say w has left-to-right maxima $x_1 < x_2, \ldots < x_i = n$ and right-to-left maxima $n = y_j > y_{j-1} > \ldots > y_1$. If we insert $n + 1$ to the left of x_k, the resulting permutation will have left-to-right maxima $x_1 < \ldots < x_{k-1} < n + 1$, and right-to-left maxima $n + 1 > n = y_j > \ldots > y_1$. If we insert $n + 1$ to the right of y_k, the resulting permutation will have left-to-right maxima $x_1 < x_2 < \ldots x_i = n < n + 1$, and right-to-left maxima $n + 1 > y_{k-1} > \ldots > y_1$.

This means that the number of children a given Baxter permutation has (and how many children those children will have, and so on) is entirely encoded by the number i of left-to-right maxima, and the number j of right-to-left maxima. Thus, the tree with root $(1,1)$, and the property that every node (i,j) has children $(1,j+1),(2,j+1),\dots(i,j+1),(i+1,j),(i+1,j-1),\dots(i+1,1)$ will be isomorphic to the generating tree for Baxter permutations.

Corollary 1.4. *Baxter permutations have the same number of descents as inverse descents.*

We already have a combinatorial rule for when we can insert a new largest entry into a Baxter permutation and still be a Baxter permutation, so now we come up with a combinatorial rule for when we can insert a new smallest entry into a Baxter permutation and still be a Baxter permutation. By inserting a new smallest entry, we mean that we increase all the labels in the existing permutation by 1, and then insert a new entry with label 1, so that if the original permutation was a standard permutation of n letters on $[n]$, then the result will be a standard permutation on $[n+1]$.

Lemma 1.5. *Inserting a new smallest label into position j into a permutation is equivalent to rotating the permutation matrix 180°, inserting n into position $n+1-j$, and then rotating the permutation matrix 180° again.*

Consequently, given a Baxter permutation w, the admissible places we can insert a new smallest label are immediately to the left of a left-to-right minima, or immediately to the right of a right-to-left minima.

Again, we only need to keep track of the number of left-to-right and right-to-left maxima. Each of these corresponds to a place where we can insert $n+1$, and then we know there will be a complementary place we can insert 1 to stay fixed under conjugation by the longest element. We know how inserting $n+1$ will affect the number of left-to-right and right-to-left maxima. Inserting 1 will in general not create any new left-to-right or right-to-left maxima, except in the case where we are adding 1 to the beginning or end of the word.

1.2 Generating Tree for Baxter permutations fixed under 90° rotation

For a permutation to be fixed under 90° rotation, it is equivalent to say that if $w_i = j$, then $w_j = n+1-i$, $w_{n+1-i} = n+1-j$, and $w_{n+1-j} = i$. If we consider the cycle structure of this permutation, in general it makes a 4-cycle $(i, j, n+1-i, n+1-j)$. If this were to degenerate into a smaller cycle, we would have that $i = n+1-i$. This forces to $n = 2i+1$ to be odd, and it also forces $i = j$, which means it actually degenerates to a single central fixed point. Thus, a permutation fixed under this action must have length $4m$ or $4m+1$.

If w is a Baxter permutation of length n with k descents, then by Corollary 1.4, w^{-1} will have k descents, and $w_0 w^{-1}$ will have $n-1-k$ descents. So for a Baxter permutation to be fixed by this action, we must have $k = n-1-k$, which implies that n must be odd, and along with our previous observation must be $n = 4m+1$. Thus, a Baxter permutation fixed under 90° rotation will consist of a single central fixed point, and four cycles of the form $(i, j, n+1-i, n+1-j)$.

In particular, for $n > 1$, such a permutation will have a four cycle of the form $(1, j, n, n+1-j)$, which means the permutation starts with j, has n in the j^{th} position, 1 in the $(n+1-j)^{th}$ position, and $n+1-j$ at the end. We already know that we can remove n and 1 from a Baxter permutation and still be a Baxter permutation. It's not hard to see that we can also remove the first element or the last element from a Baxter permutation and still be a Baxter permutation. So if we take a Baxter permutation fixed under 90° rotation and the remove the largest label, the smallest label, the first label, and the last label, then we will still have a Baxter permutation, and it will still be fixed under 90° rotation. Thus, we can create a generating tree, with the identity permutation on 1 element as the root

In order to create a four cycle, we have to come up with a combinatorial rule for when we can insert a letter at the beginning (resp. end) of a Baxter permutation, and still have it be a Baxter permutation. To insert a letter j at the beginning of a permutation w of length n, we mean that we increase all the labels greater than or equal to j in w by 1, and then prepend j, so the result is a standard permutation on $[n+1]$.

Lemma 1.6. *Inserting j at the end of a word is equivalent to rotating the permutation matrix 90° clockwise, inserting n into position $n+1-j$, and the rotating the permutation matrix 90° counter-clockwise.*

Similarly, inserting j at the beginning of a word is equivalent to rotating a permutation matrix 90° counter-clockwise, inserting n into position j, and then rotating back 90° clockwise.

Consequently, we can insert j at the end (resp. beginning) of a Baxter permutation and still have it be a Baxter permutation if and only if all entries smaller than j appear to the left (resp. right) of j, or if all entries bigger than $j-1$ appear to the right (resp. left) of $j-1$.

Note that inserting j at the end (resp. beginning) of Baxter permutation can possibly decrease the number of right-to-left (resp. left-to-right) maxima, as any previous left-to-right (resp. right-to-left) maxima that was less than j will no longer be one after j is inserted at the end (resp. beginning).

Theorem 1.7. *For a Baxter permutation fixed under 90° rotation, for every admissible position we can insert a new largest label and still have a Baxter permutation, it is also possible to insert a new smallest label, a new beginning label, and a new final label so that the result is a Baxter permutation fixed under 90° rotation.*

Now, we want analyze how doing these four insertions changes the number of left-to-right and left-to-right maxima.

Lemma 1.8. *If w is a Baxter permutation fixed under 90° rotation, then w has the same number of left-to-right and right-to-left maxima. In particular, if*

w has left-to-right maxima in positions $x_1 < x_2 < \ldots < x_j$ and right-to-left maxima at positions $y_j < y_{j-1} < \ldots < y_1$, and we do a four cycle insertion corresponding to being able to insert a new largest label to the right of w_{y_i} (or to the left of w_{x_i}), then the resulting Baxter permutation fixed under $90°$ rotation will have $i+1$ left-to-right maxima and $i+1$ right-to-left maxima.

We now have enough information to analyze the generating tree for Baxter permutations fixed under rotation by $90°$ degrees. If a Baxter permutation fixed under rotation by $90°$ degrees has $i+1$ left-to-right maxima and $i+1$ right-to-left maxima, then it will have $2i+2$ children. There will be $i+1$ children with number of left-to-right (and right-to-left) maxima being $2, 3, \ldots i+2$ corresponding to inserting a new largest label to the left of a left-to-right maxima, and $i+1$ children with number of left-to-right (and right-to-left) maxima being $2, 3, \ldots i+2$ corresponding to inserting a new largest label to the right of a right-to-left maxima.

Thus, this generating tree is almost like the Catalan tree, except each parent with label $i+1$ has two (not one) children with a label between 2 and $i+2$, and our root will have label 1. This implies that the number of elements of a given rank m must be $2^m C_m$.

1.3 Remarks

The fact that this enumeration has such an elegant closed formula means that it is likely that there is an underlying combinatorial bijection. However, as with Chung, Graham, Hoggat, and Kleiman, the method of generating trees does not make such an interpretation transparent.

Additionally, one might hope that it is possible to extend the previous "q=-1" result for Baxter permutations fixed under $180°$ rotation to an instance of the cyclic sieving phenomenon. That is to say, finding a polynomial $f(q)$ where gives an enumeration of Baxter permutations (perhaps with respect to some statistics), $f(-1)$ counts how many of these Baxter permutations are fixed under $180°$ rotation, and $f(i) = f(-i)$ counts how many of them are fixed under $90°$ rotation. However, the natural candidate of $\Theta_{k,\ell}(q)$ does not work, and it does not appear that it can be easily modified to give such a result.

References

[1] N. Bonichon, M. Bousquet-Mélou, and E. Fusy. Baxter permutations and plane bipolar orientations. *Electronic Notes in Discrete Mathematics*, 31(C):69–74, 2008.

[2] K. Dilks. Involutions on Baxter objects. *DMTCS Proceedings*, 0(01), 2012.