PATTERNS AND CYCLES IN DYNAMICAL SYSTEMS

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The permutations realized by one-dimensional dynamical systems give insight into their short-term behavior and provide a way of understanding the system's complexity. Given a map $f : X \to X$ from a linearly ordered set X to itself and a point $x \in X$, consider the finite sequence

$$x, f(x), f(f(x)), \ldots, f^{n-1}(x).$$

If these n values are different, then their relative order determines a permutation $\pi \in \mathcal{S}_n$ obtained by replacing the smallest value by a 1, the second smallest by a 2, and so on; we denote this operation by $\mathrm{st}(x, f(x), f(f(x)), \ldots, f^{n-1}(x)) = \pi$. For example, if $f(x) = \{\frac{1+\sqrt{5}}{2}x\}$, where $\{y\}$ is the fractional part of y, and $x = .52$, we have

$$\mathrm{st}(x, f(x), f^2(x), f^3(x)) = \mathrm{st}(0.52, 0.84, 0.36, 0.58) = 2413.$$

We write $\mathrm{Pat}(x, f, n) = \pi$, and say that π is an *allowed pattern* of f. If the first n iterations of f starting with x are not distinct, then $\mathrm{Pat}(x, f, n)$ is not defined. We denote the set of allowed patterns of length n by

$$\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n,$$

and let $\mathrm{Allow}(f) = \bigcup_{n \geq 1} \mathrm{Allow}_n(f)$. It was shown in [3] that if f is a piecewise monotone map of the unit interval, then the number of allowed patterns is asymptotic to k^n, where the topological entropy of f (an important measure of complexity) is equal to $\log(k)$. Following this result, permutation-based techniques have become an important way of measuring the complexity of time-series [12].

However, the question of understanding permutations in dynamical systems was introduced much earlier in the context of Sarkovskii's Theorem about periodic points of interval maps. Recall that point $x \in X$ is an n-periodic point of f if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < n$.

Theorem 1 ([10]). *If a continuous map f of the unit interval has an m-periodic point and $\ell \lhd m$ in the Sarkovskii ordering*

$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 7\cdot 2^n \lhd 5\cdot 2^n \lhd 3\cdot 2^n \lhd \cdots \lhd 7\cdot 2 \lhd 5\cdot 2 \lhd 3\cdot 2 \lhd \cdots \lhd 9 \lhd 7 \lhd 5 \lhd 3,$$

then f must also have an ℓ-periodic point.

We may refine this question to consider how the existence of periodic points in f with a certain permutation structure forces others to occur. Let $x \in X$ be a n-periodic point such that $\mathrm{Pat}(x, f, n) = \pi_1\pi_2 \ldots \pi_n$, written in one-line notation. We say that the *cycle type* of the point x is the cycle such that $\hat{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathcal{C}_n$, written in cycle notation. In particular, because the map $\pi \to \hat{\pi}$ identifies the cyclic rotations of π, the cycle type of an n-periodic point does not depend on the representative of the periodic orbit. For this reason, we will always take the representative of an n-periodic orbit to be the value x such that

$x > f^i(x)$ for all $1 \le i < n$ and, when we write $\hat{\pi}$, the permutation π is the one for which $\pi_1 = n$. We denote the set of all cycle types of the interval map f by

$$\mathrm{AlCyc}_n(f) = \{\hat{\pi} \in \mathcal{C}_n : x \text{ is an } n\text{-periodic point of } f \text{ and } \mathrm{Pat}(x, f, n) = \pi\} \subseteq \mathcal{C}_n,$$

and let $\mathrm{AlCyc}(f) = \bigcup_{n=1}^{\infty} \mathrm{AlCyc}_n(f)$; we call this set the *allowed cycles* of f.

Definition 1 ([2])**.** *Given a class of functions $\mathcal{F}$, a cycle $\hat{\pi}$ is said to force a cycle $\hat{\tau}$, written $\hat{\tau} \trianglelefteq \hat{\pi}$, if every function $f \in \mathcal{F}$ that has a periodic point of cycle type $\hat{\pi}$ also has a periodic point with cycle type $\hat{\tau}$.*

For the class of continuous functions, the forcing relation on cycles is a partial order [2]; it is shown in Figure 1 for cycles with $n \le 4$.

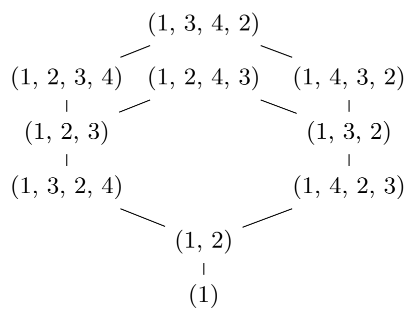

FIGURE 1. The poset of forcing relations, $n \le 4$, for continuous functions.

We will focus on two classes of maps which highlight the close relationship between properties of the dynamical system and the combinatorial structure of its allowed cycles. For $\beta > 1$, the β-shift and $-\beta$-shift, respectively, defined on the unit interval are given by

$$F_\beta(x) = \{\beta x\} \quad \text{and} \quad G_\beta(x) = 1 - \{\beta x\}.$$

The topological entropy, a measure of complexity, of both F_β and G_β is $\log(\beta)$.

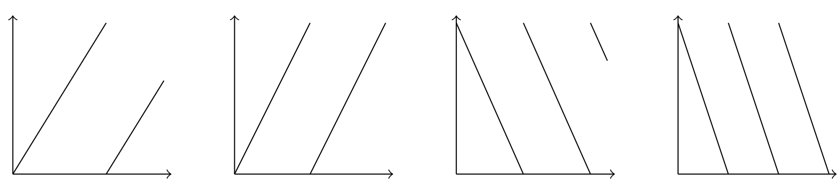

FIGURE 2. Graphs of (a) $F_\beta(x) = \{\beta x\}$ for $\beta = \frac{1+\sqrt{5}}{2}$; (b) $F_N(x) = \{Nx\}$ for $N = \lceil \beta \rceil = 2$; (c) $G_\beta(x) = 1 - \{\beta x\}$ for $\beta = \frac{9}{4}$; and (d) $G_N(x) = 1 - \{Nx\}$ and $N = \lceil \beta \rceil = 3$.

In order to view the maps F_β and G_β combinatorially, we transform each map into a shift $\Sigma(w_1 w_2 w_3 \ldots) = w_2 w_3 \ldots$ over a set of words using non-integral base expansions. Let $\mathcal{W}_\beta$ be the set of all β-expansions of $x \in [0, 1)$ where the order on words is lexicographical, see [7]. Since words $w \in \mathcal{W}_\beta$ correspond to $x \in [0, 1)$ by $f_\beta(x) = \sum_{j=0}^{\infty} \frac{w_j}{\beta^j}$, the the map F_β is equivalent to the shift map Σ_β on $(\mathcal{W}_\beta, <_{\mathrm{lex}})$.

Similarly, let $\mathcal{W}_{-\beta}$ be the set of all $-\beta$-expansions of $x \in (0,1]$, see [6]. The order on the set of words $\mathcal{W}_{-\beta}$ is the alternating order whereby $v_1v_2v_3\ldots <_{\text{alt}} w_1w_2w_3\ldots$ if there exists some i such that $v_j = w_j$ for all $j < i$ and $(-1)^i(v_i - w_i) > 0$. Since words $w \in \mathcal{W}_{-\beta}$ correspond to $x \in (0,1]$ by $g_\beta(x) = -\sum_{j=0}^{\infty} \frac{w_j+1}{(-\beta)^j}$, the map G_β is equivalent to the shift map $\Sigma_{-\beta}$ on $(\mathcal{W}_{-\beta}, <_{\text{alt}})$.

It follows that a cycle $\hat{\pi} \in \text{AlCyc}_n(F_\beta)$ if and only if there exists an n-periodic word $w = (w_1w_2\ldots w_n)^\infty \in \mathcal{W}_\beta$ such that $\text{Pat}(w, \Sigma_\beta, n) = \pi$. Likewise, $\hat{\pi} \in \text{AlCyc}_n(G_\beta)$ if and only if $\text{Pat}(w, \Sigma_{-\beta}, n) = \pi$ for some n-periodic word $w = (w_1w_2\ldots w_n)^\infty \in \mathcal{W}_{-\beta}$. The first line of the following theorem is known; the negative case can be found in [11].

Theorem 2. *If $\beta < \gamma$, then $\mathcal{W}_\beta \subset \mathcal{W}_\gamma$ (resp. $\mathcal{W}_{-\beta} \subset \mathcal{W}_{-\gamma}$) and so*

$$\text{AlCyc}_n(F_\beta) \subseteq \text{AlCyc}_n(F_\gamma) \quad \textit{and} \quad \text{AlCyc}_n(G_\beta) \subseteq \text{AlCyc}_n(G_\gamma).$$

In particular, unlike for the class of continuous maps, the forcing order for the class of β-shifts (resp. $-\beta$-shifts) is a total order.

To study the total order, we associate to each cycle a measure of the minimal complexity of a system containing the given cycle type by

$$B_p(\hat{\pi}) = \inf\{\beta : \hat{\pi} \in \text{AlCyc}(F_\beta)\} \quad \text{and} \quad \overline{B}_p(\hat{\pi}) = \inf\{\beta : \hat{\pi} \in \text{AlCyc}(G_\beta)\}.$$

We first determine the n-periodic k-ary words w such that $\text{Pat}(w, \Sigma_k, n) = \pi$ and $\text{Pat}(w, \Sigma_{-k}, n) = \pi$, respectively.

Definition 3. A k-segmentation (resp. $-k$-segmentation) of $\hat{\pi} = \hat{\pi}_1\hat{\pi}_2\ldots\hat{\pi}_n$ is a sequence $0 = e_0 \leq e_1 \leq \cdots \leq e_k = n$ such that each segment $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2}\ldots\hat{\pi}_{e_{t+1}}$ is increasing (resp. decreasing).

To a segmentation of $\hat{\pi}$, we associate the finite word $\omega = z_1z_2\ldots z_{n-1}$, defined by taking $z_i = j$ whenever $e_j < \pi_i \leq e_{j+1}$ for $1 \leq i \leq n-1$ and say that the segmentation *defines* ω.

Theorem 4 ([1]). *Let $\hat{\pi} \in \mathcal{C}_n$ and let ω be the word defined by a k-segmentation (resp. $-k$-segmentation) of $\hat{\pi}$. If ω is primitive, i.e. is not equal to one of its non-trivial cyclic rotations, then the word ω^∞ is an n-periodic word such that* $\text{Pat}(\omega^\infty, \Sigma_k, n) = \pi$ *(resp.* $\text{Pat}(\omega^\infty, \Sigma_{-k}, n) = \pi$*).*

Example 5. Consider the cycle $\hat{\pi} = (5,3,1,4,2) = 45123$, and so $\pi = 53142$. Notice that $\hat{\pi}$ has a unique 2-segmentation given by $(e_0, e_1, e_2) = (0, 2, 5)$ defining the word $\omega = 11010$. We can see that $\text{Pat}(\omega^\infty, \Sigma_2, 5) = \pi$ because taking shifts of ω^∞ we obtain

$$\text{Pat}(\omega^\infty, \Sigma_2, 5) = \text{st}((11010)^\infty, (10101)^\infty, (01011)^\infty, (10110)^\infty, (01101)^\infty) = 53142,$$

where the order on words is lexicographical. The word $\omega^\infty \in \mathcal{W}_2$ corresponds to $x = \frac{26}{31}$, the representative of the unique 5-periodic orbit of F_2 with cycle type $\hat{\pi}$.

Theorem 6 ([1]). *Let $N_p(\hat{\pi}) = \lceil B_p(\hat{\pi}) \rceil$ and $\overline{N}_p(\hat{\pi}) = \lceil \overline{B}_p(\hat{\pi}) \rceil$. Then*

$$N_p(\hat{\pi}) = 1 + \text{des}(\hat{\pi}) \quad \textit{and} \quad \overline{N}_p(\hat{\pi}) = 1 + \text{asc}(\hat{\pi}) + \delta(\hat{\pi}),$$

where $\delta(\hat{\pi}) = 1$ if the word ω defined by the $-(1+\text{asc}(\hat{\pi}))$-segmentation of $\hat{\pi}$ is not primitive; and $\delta(\hat{\pi}) = 0$ otherwise.

FIGURE 3. Cycle diagrams for (a) $\hat{\pi} = 4516723$ and (b) $\hat{\pi} = 87651432$ from Example 7. including sloped lines to indicate the choice of segmentation.

Example 7. (a) Consider the cycle $\hat{\pi} = (7, 3, 1, 4, 6, 2, 5) = 4516723$ and so $\pi = 7314625$, see Figure 3. The unique 3-segmentation of $\hat{\pi}$ is given by $(e_0, e_1, e_2, e_3) = (0, 2, 5, 7)$ defining the word $\omega = 2101201$. By Theorem 4, we have $\mathrm{Pat}(\omega^\infty, \Sigma_3, 7) = \pi$. We calculate $B_p(\hat{\pi}) = 2.3708832$ by solving for the unique $\beta > 1$ such that for $w = \omega^\infty$ we have $f_\beta(w) = 1$.

(b) Consider the cycle $\hat{\pi} = (8, 2, 7, 3, 6, 4, 5, 1) = 87651432$ and so $\pi = 82736451$, see Figure 3. The unique -2-segmentation of $\hat{\pi}$ is given by $(e_0, e_1, e_2) = (0, 5, 8)$ defining the word $\omega = 1010100$. By Theorem 4, we have $\mathrm{Pat}(\omega^\infty, \Sigma_{-2}, 8) = \pi$. We find that $\overline{B}_p(\hat{\pi}) = 1.9878$, and so G_β has an 8-periodic point with cycle type $\hat{\pi}$ if and only if $\beta > 1.9878$.

Observation: Since the number descents (resp. ascents) in cycles is asymptotically normally distributed [9], we have that $N_p(\hat{\pi})$ (resp. $\overline{N}_p(\hat{\pi})$) is also asymptotically normally distributed with mean $\frac{n+1}{2}$ and variance $\frac{n+1}{12}$.

$B(\hat{\pi})$	polynomial	$\hat{\pi}$	ω^∞
1.3803	$x^4 - x^3 - 1$	$(4,1,2,3) = 2341$	$(1000)^\infty$
1.4656	$x^3 - x^2 - 1$	$(3,1,2) = 231$	$(100)^\infty$
1.7549	$x^4 - x^3 - x^2 - 1$	$(4,3,1,2) = 2413$	$(1100)^\infty$
1.8393	$x^3 - x^2 - x - 1$	$(3,2,1) = 312$	$(110)^\infty$
1.9276	$x^4 - x^3 - x^2 - x - 1$	$(4,3,2,1) = 4123$	$(1110)^\infty$
2.2775	$x^4 - 2x^3 - x - 1$	$(4,1,3,2) = 3421$	$(2010)^\infty$
2.5214	$x^4 - 2x^3 - x^2 - 2$	$(4,2,1,3) = 3142$	$(2101)^\infty$
2.6968	$x^4 - 2x^3 - x^2 - 2x - 1$	$(4,2,3,1) = 4312$	$(2120)^\infty$

$\overline{B}(\hat{\pi})$	polynomial	$\hat{\pi}$	ω^∞
1	$x^3 - 2x^2 + x$	$(3,2,1) = 312$	$(100)^\infty$
1.4655712	$x^3 - 2x^2 + x^2 - x + 1$	$(4,2,1,3) = 3142$	$(1001)^\infty$
1.7549	$x^3 - 2x^2 + x - 1$	$(3,1,2) = 231$	$(101)^\infty$
1.754877	$x^4 - 2x^3 + x^2 - x$	$(4,2,3,1) = 4312$	$(1000)^\infty$
1.8832	$x^4 - 2x^3 + x^2 - 2x + 1$	$(4,1,3,2) = 3421$	$(1011)^\infty$
2.205569	$x^4 - 2x^3 + 2x^2 - x + 1$	$(4,3,1,2) = 2413$	$(2101)^\infty$
2.32471	$x^4 - 3x^3 + 2x^2 - x$	$(4,3,2,1) = 4123$	$(2100)^\infty$
2.80715	$x^4 - 2x^3 + x^2 - 2x + 2$	$(4,1,2,3) = 2341$	$(2012)^\infty$

TABLE 1. The values of $B_p(\hat{\pi})$ and $\overline{B}_p(\hat{\pi})$, respectively, for cycles $\hat{\pi}$ of length $n = 3, 4$, illustrating the forcing order on cycles for the β-shifts and $-\beta$-shifts.

The techniques presented here can be generalized to larger classes of maps as long as the sets of words associated to the maps are well-understood. It is worth mentioning that the forcing order for the class of logistic functions $f_r(x) = rx(1-x)$ parametrized by the value $r \in (1 + \sqrt{6}, 4]$ is not a total order.

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