On the growth rate of the centrosymmetric permutations in a class

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For a permutation $\pi \in S_n$, the *reverse–complement* of π, denoted $rc(\pi)$, is the permutation in S_n whose ith entry is $n+1-\pi(n+1-i)$. Equivalently, rc operates on a permutation by rotating its diagram by a half turn. We say π is *centrosymmetric* if $\pi = rc(\pi)$. A permutation class $\mathcal{C}$ is *rc-invariant* if $\mathcal{C} = rc(\mathcal{C})$. Let $\mathcal{C}^{rc}_{2n}$ (resp. $\mathcal{C}^{rc}$) denote the set of centrosymmetric permutations in $\mathcal{C}_{2n}$ (resp. $\mathcal{C}$).

At Permutation Patterns 2016, Alex Woo presented the following open question: *For which rc-invariant permutation classes $\mathcal{C}$ is it true that $\mathcal{C}_n$ and $\mathcal{C}^{rc}_{2n}$ have the same exponential growth rate?* In this presentation, we investigate this question.

Let $\mathrm{gr}(\mathcal{C})$ denote the growth rate of $\mathcal{C}_n$, and let $\mathrm{gr}^{rc}(\mathcal{C})$ denote the growth rate of $\mathcal{C}^{rc}_{2n}$:

$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}_n|^{1/n} \qquad \text{and} \qquad \mathrm{gr}^{rc}(\mathcal{C}) = \lim_{n\to\infty} |\mathcal{C}^{rc}_{2n}|^{1/n}.$$

(To simplify matters, we will assume these limits exist.) In this notation, the question asks to determine when $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$. A similar project is undertaken in [**4**], but with involutions rather than centrosymmetric permutations.

We have found two rc-invariant "two-by-four classes" $\mathcal{C}$ for which $\mathrm{gr}^{rc}(\mathcal{C}) < \mathrm{gr}(\mathcal{C})$:

- $\mathrm{gr}(\mathrm{Av}(4231, 1324)) = 2 + \sqrt{2}$ [**3**], but $\mathrm{gr}^{rc}(\mathrm{Av}(4231, 1324)) = 2$;
- $\mathrm{gr}(\mathrm{Av}(4321, 2143)) = (3 + \sqrt{5})/2$ [**2**], but $\mathrm{gr}^{rc}(\mathrm{Av}(4321, 2143)) = 2$.

The rest of this abstract gives more examples of classes $\mathcal{C}$ for which $\mathrm{gr}^{rc}(\mathcal{C}) \neq \mathrm{gr}(\mathcal{C})$, including two geometric grid classes; then we give some sufficient conditions for a class to satisfy $\mathrm{gr}(\mathcal{C}) = \mathrm{gr}^{rc}(\mathcal{C})$, such as being $\oplus$-closed with growth rate less than a certain real number ξ.

Examples from unions of permutation classes

If $\mathcal{D}$ is a permutation class that is not rc-invariant, then $\mathcal{D} \cap rc(\mathcal{D})$ is a proper subclass of $\mathcal{D}$, but $\mathcal{D} \cap rc(\mathcal{D})$ has the same centrosymmetric permutations as $\mathcal{D}$. If B is a basis for $\mathcal{D}$, then $B \cup rc(B)$ is a basis for $\mathcal{D} \cap rc(\mathcal{D})$, so $\mathrm{Av}(B)^{rc} = \mathrm{Av}(B \cup rc(B))^{rc}$. This is why it makes sense to restrict the scope of the question to rc-invariant classes.

For instance, Av(312) is not rc-invariant, and $\mathrm{Av}(312)^{rc} = \mathrm{Av}(312, 231)^{rc}$. Looking at the growth rates, we find that $\mathrm{gr}^{rc}(\mathrm{Av}(312)) = \mathrm{gr}^{rc}(\mathrm{Av}(312, 231)) = 2$ [**5**] and $\mathrm{gr}(\mathrm{Av}(312, 231)) = 2$ [**9**], whereas $\mathrm{gr}(\mathrm{Av}(312)) = 4$ [**7**].

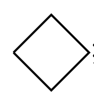
Furthermore, if $\mathcal{D}$ is not rc-invariant, then $\mathcal{D} \cup rc(\mathcal{D})$ is an rc-invariant class that properly contains $\mathcal{D}$, but $\mathcal{D} \cup rc(\mathcal{D})$ has the same centrosymmetric permutations as $\mathcal{D} \cap rc(\mathcal{D})$. We have $\mathrm{gr}(\mathcal{D} \cup rc(\mathcal{D})) = \mathrm{gr}(\mathcal{D})$, but $\mathrm{gr}^{rc}(\mathcal{D} \cup rc(\mathcal{D})) = \mathrm{gr}^{rc}(\mathcal{D} \cap rc(\mathcal{D}))$. We can use this idea to find many rc-invariant classes of the form $\mathcal{C} = \mathcal{D} \cup rc(\mathcal{D})$ for which $\mathrm{gr}^{rc}(\mathcal{C}) < \mathrm{gr}(\mathcal{C})$:

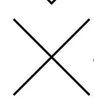
- If $\mathcal{C} = \mathrm{Av}(312) \cup \mathrm{Av}(231)$, then $\mathrm{gr}(\mathcal{C}) = \mathrm{gr}(\mathrm{Av}(312)) = 4$ [**7**]; however, $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}^{rc}(\mathrm{Av}(312, 231)) = 2$ [**5**].
- If $\mathcal{C} = \mathrm{Av}(4123) \cup \mathrm{Av}(2341)$, then $\mathrm{gr}(\mathcal{C}) = \mathrm{gr}(\mathrm{Av}(4123)) = 9$ [**10**], but $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}^{rc}(\mathrm{Av}(4123, 2341)) = 4$.
- If $\mathcal{C} = \mathrm{Av}(4312) \cup \mathrm{Av}(3421)$, then $\mathrm{gr}(\mathcal{C}) = \mathrm{gr}(\mathrm{Av}(4312)) = 9$ [**12**], but $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}^{rc}(4312, 3421) = 2 + \sqrt{5}$.

Examples from geometric grid classes

Seeing the examples above, one may hope that this property, namely that $\mathcal{C}$ has a proper subclass $\mathcal{D}$ for which $\mathcal{C} = \mathcal{D} \cup rc(\mathcal{D})$, is the only obstruction from satisfying $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$. However, this is not the case, as the following examples show.

Let M be a $\{0, 1, -1\}$-matrix. The *standard figure* of M is the set of line segments determined by the entries of M in the following way: if the entry is 1, replace the entry with a line segment with positive slope; if the entry is -1, replace it with a line segment with negative slope; if the entry is 0, put nothing there. For example,

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ has standard figure ;

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ has standard figure .

Let the *geometric grid class* of M, denoted $\mathrm{Geom}(M)$, be the set of permutations that can be plotted on the standard figure of M: explicitly, $\mathrm{Geom}(M)$ is the set of permutations π such that there is a set of points $(x_1, y_1), \ldots, (x_n, y_n)$ lying on the standard figure of M such that the list $x_1, \ldots, x_n$ is in increasing order while the list $y_1, \ldots, y_n$ has the same relative order as $\pi(1), \ldots, \pi(n)$. Note that $\mathrm{Geom}(M)$ is a class.

Let $\mathcal{C}$ be one of the following two geometric grid classes:

$$\mathcal{C} = \mathrm{Geom}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad \mathcal{C} = \mathrm{Geom}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Equivalently, $\mathcal{C}$ is the class of permutations that can be drawn on a circle (resp. an X). Then $\mathcal{C}$ is rc-invariant, and $\mathcal{C}$ has no proper subclass $\mathcal{D}$ such that $\mathcal{C} = \mathcal{D} \cup rc(\mathcal{D})$ — in fact, every geometric grid class is *atomic* [**1**] (meaning it is not a finite union of proper subclasses). Furthermore, $\mathcal{C}$ is generated by its centrosymmetric elements: for each $\pi \in \mathcal{C}$, there is $\rho \in \mathcal{C}^{rc}$ that contains π. However, $\mathrm{gr}(\mathcal{C}) = 2 + \sqrt{2}$ (see [**11**] and [**6**]) and $\mathrm{gr}^{rc}(\mathcal{C}) = 2$.

Theorem on geometric grid classes

Let M be a $\{0, 1, -1\}$-matrix. The *cell graph* of M is the graph whose vertices are the *non-zero* entries of M, where two entries are adjacent if (1) they share a row or column and (2) there are no non-zero entries between them in their row or column. We say M is a *forest* if its cell graph is a forest. Geometric grid classes of forests are discussed in [**1,** Sec. 3]. We have proved the following result about such classes.

Theorem 1. If M is a centrosymmetric matrix that is a forest, then $\mathrm{gr}^{rc}(\mathrm{Geom}(M)) = \mathrm{gr}(\mathrm{Geom}(M))$.

Note that the grid matrices for the circle class and the X class are not forests (since the four non-zero entries are in a cycle), so this theorem does not apply to them.

Results on $\oplus$-closed or $\ominus$-closed classes

Recall that $\mathcal{C}$ is said to be $\oplus$-*closed (direct sum–closed)* when it satisfies the following property: if $\pi, \rho \in \mathcal{C}$, then $\pi \oplus \rho \in \mathcal{C}$. The definition of $\ominus$-*closed (skew sum–closed)* is similar. Note that every $\oplus$-closed or $\ominus$-closed class is atomic.

Since an $\oplus$-closed class can be transformed to a $\ominus$-closed class by taking the reverse of the permutations in the class (and rc commutes with reversing), the following statements for $\oplus$-closed classes will also apply to $\ominus$-closed classes, but we will only state them for $\oplus$-closed classes.

Of the several examples $\mathcal{C}$ we have found for which $\mathrm{gr}^{rc}(\mathcal{C}) \neq \mathrm{gr}(\mathcal{C})$, none of them has $\mathrm{gr}^{rc}(\mathcal{C}) > \mathrm{gr}(\mathcal{C})$, and none of them is $\oplus$-closed. This leads us to the following conjecture:

Conjecture 2. (a) For any class $\mathcal{C}$, $\mathrm{gr}^{rc}(\mathcal{C}) \leq \mathrm{gr}(\mathcal{C})$. **(b)** If $\mathcal{C}$ is rc-invariant and $\oplus$-closed, then $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$.

The next two results are partial progress towards proving part (b) of the conjecture.

Proposition 3. If $\mathcal{C}$ is rc-invariant and either $\oplus$-closed or $\ominus$-closed, then $|\mathcal{C}^{rc}_{2n}| \geq |\mathcal{C}_n|$, which implies $\mathrm{gr}^{rc}(\mathcal{C}) \geq \mathrm{gr}(\mathcal{C})$.

Proof. If $\mathcal{C}$ is rc-invariant and $\oplus$-closed, then the function $\pi \mapsto \pi \oplus rc(\pi)$ is an injection from $\mathcal{C}_n$ into $\mathcal{C}^{rc}_{2n}$. ■

If part (a) of the conjecture is true, then Proposition 3 implies part (b) of the conjecture.

Let $\widetilde{\mathcal{C}_n}$ (resp. $\widetilde{\mathcal{C}}$) denote the set of $\oplus$-indecomposable permutations in $\mathcal{C}_n$ (resp. $\mathcal{C}$), and let $\widetilde{\mathcal{C}}^{rc}_{2n}$ (resp. $\widetilde{C}^{rc}$) denote the set of centrosymmetric permutations in $\widetilde{\mathcal{C}}_{2n}$ (resp. $\widetilde{\mathcal{C}}$). Let $\mathrm{gr}(\widetilde{\mathcal{C}})$ (resp. $\mathrm{gr}^{rc}(\widetilde{\mathcal{C}})$) denote the growth rate of $\widetilde{\mathcal{C}}_n$ (resp. $\widetilde{\mathcal{C}}^{rc}_{2n}$):

$$\mathrm{gr}(\widetilde{\mathcal{C}}) = \lim_{n\to\infty} \left|\widetilde{\mathcal{C}}_n\right|^{1/n} \qquad \text{and} \qquad \mathrm{gr}^{rc}(\widetilde{\mathcal{C}}) = \lim_{n\to\infty} \left|\widetilde{\mathcal{C}}^{rc}_{2n}\right|^{1/n}.$$

Theorem 4. Let $\mathcal{C}$ be rc-invariant and $\oplus$-closed, and let $\xi \approx 2.31$ be the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$. If $\mathrm{gr}(\mathcal{C}) \le \xi$, then $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$. More precisely, each of the following statements implies the next one:

I. $\mathrm{gr}(\mathcal{C}) \le \xi$; II. $|\widetilde{\mathcal{C}}_n|$ is bounded; III. $\mathrm{gr}(\widetilde{C})$ is either 0 or 1;
IV. $\mathrm{gr}^{rc}(\widetilde{\mathcal{C}}) \le \mathrm{gr}(\mathcal{C})$; V. $\mathrm{gr}^{rc}(\mathcal{C}) = \mathrm{gr}(\mathcal{C})$.

The implication I $\Rightarrow$ II follows from the results in [**8,** Sec. 5 & 6] on sequences of $\oplus$-indecomposable permutations. The implications II $\Rightarrow$ III and III $\Rightarrow$ IV are easy.

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