Juxtaposing Catalan permutation classes with monotone ones

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This talk describes a clean and unified way to enumerate all juxtaposition classes of the form "Av(abc) next to Av(xy)", where abc is a permutation of length three and xy is a permutation of length two. The main tools are Dyck paths, decorated by sequences of points, and context free grammars, used afterwards to enumerate these decorated Dyck paths.

Juxtapositions are a simple special case of permutation grid classes. Grid classes found application mainly as tools to study the structure of other permutation classes. In [11], Vatter used the structural insights that monotone grid classes offer to classify growth rates of small permutation classes. Apart from enumerating permutation classes, several other applications of grid classes exist, among them [2, 1, 7, 8, 10, 12].

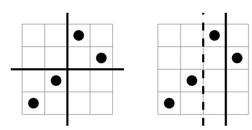

Figure 1: On the left is the unique gridding of 1243 by the gridding matrix $M = \begin{pmatrix} \emptyset & \mathrm{Av}(12) \\ \mathrm{Av}(21) & \emptyset \end{pmatrix}$. On the right are the two griddings of 1243 by $M = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(12) \end{pmatrix}$.

Each permutation in a *grid class* can be drawn into a grid so that the subpermutation in each box is in the class specified by the corresponding cell in a *gridding matrix*. See Figure 1 for an example of permutations from a *monotone grid class* (where each cell in the gridding matrix is a 21-avoider, 12-avoider, or empty). Because of their more general applicability, the study of grid classes in their own right has emerged in a few directions. For instance, it is conjectured that all monotone grid classes are finitely based, but this is only known for a few special cases, most notably those whose row-column graph is acyclic [1], and a few other special cases (see [3, 4, 13, 6]). In another direction, the role of grid classes with respect to partial well-ordering has been explored in e.g. [8, 10, 12]. Finally, while the asymptotic enumeration of monotone grid classes was answered completely by Bevan [5], exact enumeration is harder, primarily due to the difficulty of handling multiple griddings: that is, enumerating 'griddable' objects rather than 'gridded' ones. One general result here is that all geometric grid classes have rational generating functions [1], but the move from 'gridded' to 'griddable' is nonconstructive, instead relying on properties of regular languages.

As a first step towards enumerating more general grid classes, in this paper we replace one cell in the gridding matrix M of a monotone grid class by a *Catalan class*, that is, one avoiding a single permutation of length 3. For simplicity, we restrict our attention to 1×2 grids, although the techniques presented here could be used in larger grids. These 1×2 grid classes are also referred to as *juxtapositions* — in our case a Catalan class in the left cell, and a monotone class in the right one. The main result of this talk is summarised in Table 1.

$\mathrm{Av}(213\mid 21)$, $\underline{\mathbf{Av(231\mid 12)}}$	$\xleftrightarrow{\theta}$	$\mathrm{Av}(123\mid 21)$, $\underline{\mathrm{Av}(321\mid 12)}$
$\mathrm{Av}(123\mid 12)$, $\underline{\mathbf{Av(321\mid 21)}}$	$\xleftrightarrow{\psi}$	$\mathrm{Av}(213\mid 12)$, $\underline{\mathrm{Av}(231\mid 21)}$
$\mathrm{Av}(132\mid 12)$, $\underline{\mathbf{Av(312\mid 21)}}$	$\xleftrightarrow{\phi}$	$\mathrm{Av}(132\mid 21)$, $\underline{\mathrm{Av}(312\mid 12)}$

Table 1: Each row contains equinumerous classes. Classes in pairs (separated by commas) are equinumerous by symmetry. Left column and right column (of the same row) are equinumerous by one of the bijections θ, ψ, ϕ. Each bijection describes a correspondence between the underlined classes in the given row. The classes in bold are enumerated via context-free grammars.

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